GRADE 10 MATHEMATICS

Chapter 1. Numbers and scientific notation

Natural numbers:

They are set of numbers EXCEPT zero and represented by N. For instance, N={1, 2, 3, 4, 5, 6, 7, 8 ,….} is a set of natural numbers. Natural numbers could be EVEN numbers, numbers that ARE divisible by two,   such as N={2, 4, 6, 8, 10,…} or ODD numbers, numbers that are NOT divisible by two, such as N={9, 11, 13, 15, 17, 19,…}

Whole numbers:

The set of numbers that includes even numbers, odd numbers, AND zero represented by the symbol W, such as the set N={0, 13, 14, 15, 16, 17, 18, …}.

Integers:

Integers are set of whole numbers and negative numbers represented by the symbol Z, such as the set Z={-8-7 , -6, -5, -4-3,-2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, …}. Integers are either prime number, numbers that have two factors (one and itself) such as the set Z={-3, -5, -7, 3, 5, 7, 11,13, …} or composite number, numbers that have two factors (divisible by more than two numbers) such as Z={-6, -4, 4, 6, 8, 10, 12,…}. One is not a prime number.

Addition / subtraction of signed integers

1. Same signs: add and keep the sign

2. Different signs: subtract and the sign of the integer with greater absolute value

Example: 9+8=17, -9-8=--17, 9-8=1, -9+8=-1

Multiplication/ division of signed integers:

1. Same sign: multiply or divide and then keep the sign positive for the product or

the quotient

2. Different signs: multiply or divide and keep the negative sign for the product or the quotient

Example: (10) ×(5)=50, (-10) ×(-5)=50 (-10) ×(5)=-50 , (10)÷(5)=2, (-10)÷(-5)=2, (-10)÷(5)=-2

Rational numbers:

The set of numbers that can be written as fraction and represented by the symbol Q. Fractions are numbers in the form of N(numerator)/ D(denominator) where N and D can be negative or positive numbers and D is NOT zero. For example, -4/7, -3/5, -3/4, 7/5, and 9/4 are rational numbers. Numbers like √2, √3, and π that cannot be expressed as fraction are called irrational numbers. Irrational numbers written as decimal do not have finite digit terminating the number or repeating digit after decimal point. For example, the decimal representation of the number "π" starts with 3.14159265358979, but no finite number of digits can represent "π" exactly, nor does it repeat.

Real numbers:

Real numbers include all the rational and irrational numbers and represented by R.

The relationship between different types of numbers is expressed as, set is a subset of N, set N is a subset of W, W is a subset of Z, Z is a subset of Q, and Q is a subset of R.

Greatest common factor (GCF):

Greatest common factor (GCF) or divisor (GCD) for two or more integers is the largest positive number that divides into two or more integers. For example, the greatest common factor of 8 and 12 is 4. To compute the gcf for 48 and 12, we list all possible factors for integers, numbers that divide integers without leaving remainder, and then choose the largest factor common for all integers as underlined below.

Factors for 12={1, 2, 3,4, 6, <u>12</u>}

Factors for 48={1, 2, 3, 4, 6, 8, <u>12</u>, 16, 24}

Least common multiple (LCM).

Least or lowest common multiple for two or more integers is the smallest number that is divisible by all integers without leaving remainder. For example, the least common multiple for 6, 12,24, and 48 is computed and underlined below.

Multiples for 6= {6, 12, 18, 24, 30, 36, <u>48</u>, …}

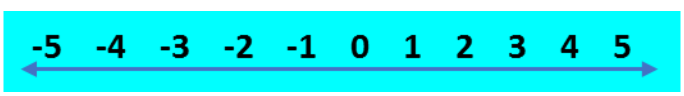
Multiples for 12= {12, 24, 36, <u>48</u>, 60, …}

Multiples for 24= {24, <u>48</u>, 72, 96, …}

Multiples for 48= {<u>48</u>, 96, 148, ….}

Absolute value:

Absolute value means how far is the number from zero on a number line as shown below. In practice, it means removing negative sign and putting the number within the sign "I I". For instance, I3I=3, I-3I=3 , I4I=4, I-4I=4, I5I=5, I-5I=5



Identity of an operation:

It is an action or quantity that when manipulated remains unchanged. For example, zero is the identity for addition and subtraction and it is called additive identity. Also, one is multiplicative identity for multiplication and division.

Example 1: Additive identity

3+5+9=3+5+9+0

47-12=47-12-0

Example 2: Multiplicative identity

9×8×7=9×8×7×1

--12×13=-12×13×1

Inverse of an operation:

Inverse operations are opposite operations as addition is opposite of subtraction and division is opposite of multiplication. It is an action which when is added to a quantity or number it will make it zero and when multiplied by a quantity or a number it will make it one. For example, inverse of -8 is 8 which when added to -8, that is, -8+8=0. Example for inverse operation of -8 in multiplication is 1/-8 which is equal to 1 multiplied together.

Scientific notation:

Mathematicians, engineers, and scientists use scientific notation in computing and expressing numbers that are too large or too small. Think about numbers such as (2,000,000,000) × (3,000000000000). Isn't it easier and simpler to write it as (2×10^9) × (3×10^{12}) instead of spending time counting zeros? Which way has greater chance of mistake? Place a decimal after the first non-zero digit and then count the number of digits and zeros after the first digit and use as the exponent of the base 10. The exponent will have a negative or a positive sign, if the decimal point is moved to the right or to the left of the first digit, respectively. It can be generalized as b×10^n to convert decimal notation to scientific notation where "b" ,1≥b≥9, is any real number called **significand**, and "n" is an integer. If necessary, we can change the scientific notation to decimal or normal notation by moving the decimal point to the right if the exponent is positive and to the left if the exponent is negative. Practice the following examples.

Decimal to scientific notation:

Example 1. 400=4×10^2

Example 2. 52387=5.2387×10^4

Example 3. 0.3=3×10^{-1}

Example 4. -756,000=-7.56×10^{5}

Example 5. 0.0000000000896=8.96×10^{-11}

Scientific notation to decimal notation:

Example 1: 6×10^{-1} =0.6

Example 2: 3.84 ×10^{-7}=0.000000384

Example 3: 8×$10^{9} = 8{,}000000000$

Example 4: 5.365$10^{5}$× =536500

Addition and subtraction of scientific notation:

Only like terms can be added or subtracted in exponential expressions. In scientific notation, like terms must have the same base and the same power. Otherwise, the terms should be modified to like terms.

Example 1. (7.45 ×10^{3}) $+ (4.55 \times 10^{3}$), factor out the 10^{3} factor.

10^{3} (7.45+4.55)=12× 10^{3}.= 1.2× 10^{4}

Example 2. (9.65×10^{5}) $+(9.65 \times 10^{5}$), 10^{5} (9.65+8.75)=18.4× 10^{5}.= 1.84× 10^{6}

Example 3. (5.7×10^{7})-(6.8×10^{5}) , rewrite as (5.7×10^{5}×10^{2})-(6.8×10^{5}) , factor out 10^{5},

10^{5}(570-6.8)=10^{5}(563.2)=5.632 × 10^{7}

Example 4. (3.754×10^{9}) –(2.34×10^{9}), factor out 10^{9}, 10^{9} .(375-2.34)= 1.41×10^{9}

Multiplication and division of scientific notation:

Example 1: (2.5×10^{-4}) (5×10^{7}), based on commutative and associative properties we rearrange as (2.5×5) (10^{-4} ×10^{7}), simplify it.

(12.5)(10^{3}) =1.25 × 10^{4}

Example 2: (-8.557×10^{5}) (3×10^{9}),rewrite as (-8.557×3)(10^{5} ×10^{9}),

-25671 × 10^{14}=-2.5671 × 10^{15}

Example 3: (4.5×10^{-3})÷((1.5×10^{4})(8.43×10^{5})), rewrite as

Rearrange as: $\frac{4.5\times10^{-3}}{(1.5\times8.43)(10^{4}\times10^{5})}$

simplify: $\frac{3\times10^{-3}}{(8.43)(10^{9})}$

Divide 3 by 8.43 and multiply by 10^{-12}:

0.3559×10^{-12}=3.559×10^{-13}

Example 4: $\frac{(7.5\times10^{8}}{2.5\times10^{5}}$ =3×103

Chapter 2) How to simplify radical expressions

Introduction:

Radical expression is any expression containing a radical symbol (√) and can be used to describe square root, cube root, fourth root and higher.

Examples: √49, √81, √x^2, √$(a+b)^4$.

Product and quotient rule for radicals:

Product rule. For any natural number root (square root, cubic root and higher) of real numbers √x and √y, the product is:

√x × √y = √(x×y) = √(xy) , the product of radicals is equal to the radical of the product

Example:

1. $\sqrt{7} \times \sqrt{8} = \sqrt{(7 \times 8)} = \sqrt{56}$
2. $\sqrt{a^2} \times \sqrt{a^3} = \sqrt{(a^2 \times a^3)} = \sqrt{a^5}$

Quotient rule. If $\sqrt{x}$ and $\sqrt{y}$ are real numbers of the natural number root (square root, cubic root or higher) and $y \neq 0$, then:

$\sqrt{x} / \sqrt{y} = \sqrt{(x/y)}$, the radical of a quotient is the quotient of the radicals

Example:

1. $\sqrt{50} / \sqrt{2} = \sqrt{(50/2)} = \sqrt{25} = \sqrt{5^2} = 5$ (if square root)
2. $\sqrt{6} / \sqrt{8} = \sqrt{(6/8)} = \sqrt{(3/4)}$

Simplifying roots of numbers:

1. Simplify $\sqrt{50}$. Find the largest perfect square that divides into 50. $50=25 \times 2$. Write 50 as a product of 25.

Then, use the product rule.

$\sqrt{50} = \sqrt{(25 \times 2)} = \sqrt{25} \times \sqrt{2} = 5\sqrt{2}$

2. $\sqrt{45} = \sqrt{(9 \times 5)} = \sqrt{9} \times \sqrt{5} = \sqrt{3^2} \times \sqrt{5} = 3\sqrt{5}$

Simplifying radicals containing variable:

All variables must be positive real numbers.

1. $\sqrt{(49x^3y^2)} = \sqrt{49} \times \sqrt{x^3} \times \sqrt{y^2} = \sqrt{7^2} \times \sqrt{(x^2 \times x)} \times y = 7y\sqrt{x}$
2. $\sqrt{(8a^4b^6)} = \sqrt{2^2} \times \sqrt{2} \times \sqrt{a^4} \times \sqrt{b^6} = 2\sqrt{2}\, a^2b^3$

Chapter 3) Roots, and Division and multiplication of radicals

In the expression $a^n=b$, a is the n^{th} root of number b. For n=2 and n=3, the n^{th} roots are called square root and cubic root, respectively.

Examples:

1. In $(7)^2 = 49$, 7 is the square root of 49.
2. In $(-7)^2 = 49$, -7 is the square root of 49.
3. In (4)3 =64, 4 is the cubic root of 64.

Since $(2)^6 = 64$ and $(-2)^6 = 64$, the sixth roots of 64 are +2 and -2.

Remember, for n even and b positive, there are two nth root of b.

Division and multiplication of radicals

When multiplying expressions containing radicals, use the rule $\sqrt[n]{x} \times \sqrt[n]{y} = \sqrt[n]{xy}$ together with normal procedure of multiplication.

Examples:

1. $\sqrt{5} \times \sqrt{7} = \sqrt{(5 \times 7)} = \sqrt{35}$
2. $\sqrt[5]{64} \times \sqrt[5]{2} = \sqrt[5]{64 \times 2} = \sqrt[5]{32 \times 2 \times 2} = 2\sqrt[5]{4}$
3. $\sqrt{(3x^5y^7)} \times \sqrt{(4x^3y^3)} = \sqrt{(12x^8y^{10})} = 2\sqrt{3}\, x^4y^5$
4. $(4+\sqrt{3})^2= (16+8\sqrt{3}+3) =19+\sqrt{3}$
5. $(\sqrt{x}+\sqrt{y})^2= x+2\sqrt{xy}+y$
6. $(\sqrt{x} +\sqrt{y})(\sqrt{x} -\sqrt{y})= x-y$

Division of radicals, also called rationalization of denominator (removal of all of the irrational numbers in the denominator of the fraction). Difference of the squares formula, $(a+b)(a-b)=a^2-b^2$, is used to rationalize denominators.

Examples:

1. $3/(\sqrt{3} +\sqrt{5})$, multiply the numerator and the denominator of the fraction by the

conjugate of (√3 +√5) which is (√3 -√5

3/(√3 +√5) ×(√3 -√5)./(√3 -√5) = 3 (√3 -√5) / (3-5)= 3 (√3 -√5) /-2

Rationalization of denominator for radicals:

Monomial denominator:

Example: 6√5 / 5√3

Multiply the numerator and denominator by the radical in the denominator and then simplify as needed:

(6√5 / 5√3) × (√3 / √3) = (6√5 × √3)/ (5√3 × √3) = 6√15 / 15 = 2√15 / 5

Binomial denominator:

Example: 2√3 / (5-√6)

Multiply the fraction by the conjugate of the denominator and simplify as needed.

2√3 / (5-√6) = 2√3 (5+√6) / (5-√6) (5+√6) = (10√3 + 2√18) / (25-6) = (10√3 + 6√2) /19

Reciprocals of a set of terns containing radical:

Examples: 9-2√3

The reciprocal is $\frac{1}{9-2\sqrt{3}}$

Rationalize the denominator using the method in fraction with monomial or binomial denominator and simplify.

$$\frac{1}{9-2\sqrt{3}} \times \frac{9+2\sqrt{3}}{9+2\sqrt{3}} = \frac{9+2\sqrt{3}}{(9-2\sqrt{3})(9+2\sqrt{3})} = \frac{9+2\sqrt{3}}{81-12} = \frac{9+2\sqrt{3}}{69}$$

Denominator with radicals of cubic root or higher:

Example: $9/\sqrt[4]{5}$

In this case, rewrite the denominator in terms of exponent, multiply the numerator and denominator by something that makes the exponent in the denominator 1, and then simplify as needed:

$9/\sqrt[4]{5}$ = 9/ $5^{1/4}$ =9×$5^{3/4}$/ $5^{1/4}$ ×$5^{3/4}$ = 9×$5^{3/4}$/ 5 = 9 $\sqrt[4]{125}$ /5

Chapter 4) Introduction to algebra

Algebraic expressions:

Algebraic expression is an expression made of integer constants, variables (a symbol like x that can represent different values in an expression), and algebraic operations, that is, addition, subtraction, multiplication, division, and exponentiation by a rational number. For instance, $5x^2$+9x+45,, and $\sqrt{(7-4x)/13+5x}$ are examples of algebraic expressions.

Adding and subtracting algebraic expressions:

In algebraic expression, a term is variable (letter), integer (constant or factor), or a combination of a number and variable multiplied together. For example, in 5x-3, 5x is a term with factor 5 which is called coefficient of variable x, and 3 is constant.

Addition and subtraction is only done for like terms, I.e. terms with the same letter and raised to the same power.

Example:

1. 7x+4y-3x-y=4x3y

2. $2x^2$-8y+7x+x^2 -2y-9x+5= $3x^2$-2x-10y+5, variables should be presented in alphabetical order.

Chapter 5) Writing and solving equations

Algebraic equations are one-step, two-step, or multi-step equations. Remember the goal is to have variables on one side of the equal sign with a coefficient of one and numbers on the other side of the equal sign. One-step equations are solved using only one step, two-step equations are solved using two steps, and multi-steps equations are solved using multiple steps.

Example for one-step equation:

1. a-15.5=8, solve it by adding 15.5 to both sides of the equation, a-15.5.+15.5=8+15.5, then a=23.5

2. -5.3b=15.9, solve it by dividing both sides by -5.3, .-5.3b/-5.3=15.9/-53, then b=-3.3

3. m÷4.6=5, solve by multiplying both sides by 4.6, then (m÷4.6) ×4.6=5× 4.6, and m=20.23

4. y+2/9=-5/9, solve by subtracting both by 2/9, y+2/9-2/9=-5/9-2/9, then y=-7/9

5. c-1/4=3/16, solve by adding ¼ to both sides of the equation, c-1/4+1/4=3/16+1/4, c=716

6. x+3.4=9, solve by subtracting both sides by 3.4, x+3.4-3.4=9-3.4, x=4.6

7. 7.5d=-31, solve by dividing both sides by 7.5, 7.5d/7.5=-31/7.5, and then d=-31/7.5, d=- 310/75

8. p÷9=36, solve by multiplying both sides by 9, (p÷9) ×9=36×9, p=324

9. x+2/9=-4/9, solve by subtracting both sides by 2/9, x+2/9-2/9=-4/9-2/9, and then x=-6/9, x=-2/3

10. 4/9y=3/7, solve by dividing both sides by 4/9, y=3/7 ×9/4, y=27/28, remember that we have multiplied 3/7 by reciprocal of 4/9 on the right side.

Example for two-step equations:

1. 7x-14=49, since first eliminate 14 by adding 14 to both sides of the equation:

7x-14+14=49+14

7x=63, now since variable is not by itself, both sides of the equation

must be divided by 7.

7x/7=63/7, then x=9

2. 3y+12=36, 3y+12-12=36+12, 3y=24, y=8

3. $\frac{4}{5}$ x+3=$\frac{7}{5}$, $\frac{4}{5}$ x+3-3=$\frac{7}{5}$ -3,, $\frac{4}{5}$ x=$\frac{22}{5}$, x=$\frac{22}{4}$, x=$\frac{11}{2}$, remember that both sides are divided by $\frac{4}{5}$ to eliminate variable coefficient and when you divide a fraction by another fraction, we have to multiply the numerator by the reciprocal of the denominator.

4. $\frac{1}{2}$a+$\frac{3}{4}$ = $\frac{5}{7}$, $\frac{1}{2}$a +$\frac{3}{4}$ — $\frac{3}{4}$ = $\frac{5}{7}$ — $\frac{3}{4}$, $\frac{1}{2}$a=- $\frac{1}{28}$, a= $\frac{1}{28}$ ×$\frac{2}{1}$, a=$\frac{1}{14}$

Multi step equations:

Here we concern ourselves with linear multi step equations. We make use of one—step and two-step equation strategies to solve multi-step equations.

Example:

1. 7x+5=2x-9, move variables to one side of the equal sign and the numbers to the other side off the equal sign.

7x-2x=-5-9, 5x=-14, x=-$\frac{14}{5}$

2. $\frac{3}{4}$ y+-$\frac{1}{2}$ =- $\frac{1}{3}$ y+$\frac{5}{7}$, $\frac{3}{4}$ y+ $\frac{1}{3}$ y=$\frac{5}{7}$ - $\frac{1}{2}$, $\frac{17}{12}$ y = $\frac{3}{14}$,y=$\frac{3}{14}$ ×$\frac{12}{17}$,y=$\frac{18}{119}$

Equations with variables on both sides of an equal sign:

Move variables terms to one side of the equal sign and the constant terms to the other side of the equal sign. The signs for terms moved to opposite side must be changed to opposite sign, I.e. -sign to + sign and + sign to – sign. Add/subtract like terms and continue as one-step and two-step equations.

Example:

1. 3x+9=x-1, 3x-x=-3-1, 2x-4, x=2

2. 5y-7=3-2y, 5y+2y=3+7, 7y=10, y=$\frac{10}{7}$

3. 7-4a=a-4, 7+4=a+4a, 11=5a, a=$\frac{11}{5}$

4. $\frac{3}{4}$ b --$\frac{4}{7}$ =3b+1, $\frac{3}{4}$ b-3b=$\frac{4}{7}$ +1, -$\frac{9}{4}$ b=$\frac{11}{7}$, b=$\frac{11}{7}$ ×$\frac{4}{9}$, b=$\frac{44}{63}$

Chapter 6) Ratio, rate and proportion

Ratio:

A comparison of two or more numbers or quantities of the same type, measurement, and unit is called ratio. A ratio is shown by colon (:) between numbers. As in fraction ratio needs to be simplified, find a number that divides into all numbers, start with smallest number in the ratio. If one of the numbers in the ratio becomes equal to one after being simplified, then the ratio is called unit ratio.

Example: simplify the following ratios and identify the unit ratios.

1. 12:8, 3:4, simplified
2. 10:25, 2:5, simplified
3. 9:6, 3:2, simplified
4. 12:14:22, 6:7:11, simplified
5. 12:48, 1:4, simplified to unit ratio
6. 7:21, 1:3, simplified to unit ratio

7. Lisa prepares fruit salad for a restaurant using strawberry, orange, and apple keeping a ratio of 1:2:3, respectively. She bought 15 pounds apple for Sunday dinner, how many pounds of strawberry and orange does she need for preparing salad?

15 lb÷3=5 lb Strawberry

15lb÷2=7.5 lb orange

Rate:

Like ratio, rate is a comparison of two numbers and is shown by colon (:) between numbers. The difference is that the unit is different for numbers. When we calculate we always write it in unit rate by dividing the first number by the second number. As in ratio, the order of numbers is important rate, R 3:2 is not equal to R 2:3. For instance, speed limit is 60 miles per hour which is written as R 60:1hr.

Example:

1. A truck driver travels 450 miles distance between two cities in 9 hours. What is the speed?

R 450:9 hours=R 50:1 hour=R 50/hour

2. Jack purchased a bag of 20 pounds potato $8. What is the unit cost per pound of potato?

R 8:20 lb=R 0.4:1lb=R 0.4/lb

Proportion:

A comparison of two ratios is called proportion. It is a special form of equation in algebra. To solve a proportion, first I need to change the proportion in colon form to fractional form. For example, the proportion x:2=3:4 needs to be changed to $\frac{x}{2} = \frac{3}{4}$.

Example: Solve the following proportions.

1. 2:x=5:4, $\frac{2}{x} = \frac{5}{4}$, 5x=8, x= $\frac{8}{5}$

2. Y:3=1:6, $\frac{y}{3} = \frac{1}{6}$, 6y=3, y= $\frac{3}{6}$, y= $\frac{1}{2}$

3. 7:a=4:8, $\frac{7}{a} = \frac{4}{8}$, 4a=56, a=$\frac{56}{4}$ = 14

4. 2:5=b:4, $\frac{2}{5} = \frac{b}{4}$, 5b=8, b=$\frac{8}{5}$

5. (X+3):2=7:4, $\frac{x+3}{2}$ = $\frac{7}{4}$, 4(x+3)=14, divide both by 4, x+3=$\frac{14}{4}$, subtract both sides by -3 and simplify $\frac{14}{4}$, x=$\frac{7}{2}$ -3, x= $\frac{1}{2}$

6. 4:(y-3)=2: (y+1), $\frac{4}{y-3} = \frac{2}{y+1}$, 2(y-3)=4(y+1), divide both sides by 2, y-3=2(y+1), apply distributive operation, y-3=2y+2, move variables to one side of the equal sign and the numbers to the other side, -3-2=2y-y, y=-5

7. 3: (a-2)=3:5, $\frac{3}{a-2} = \frac{3}{5}$, 3(a-2)=15, divide both sides by 3; a-2=5, add +2 to both sides, a=7

8. 5(2b+3):6=(b-2):3, $\frac{5(2b+3)}{6}$ = $\frac{b-2}{3}$, 15(2b+3)=6(b-2), divide both sides by 3, 5(2b+3)=2(b-2), apply distributive operation, 10b+15=2b-4, move variables to one side of the equal sign and the numbers the side of the equal sign, 10b-2b=-4—15, 8b=-19, b=- -$\frac{19}{8}$

Chapter 7) Percent and probability

Percent increase and decrease:

When a value or a quantity increases, we can compute the increase as demonstrated by following example.

Example:

Margaret's stock has increased from $85 per share to $130 per share, recently. How much is the percent increase per share?

Method 1: $\frac{new\ value-original\ value}{original\ value}$ = $\frac{130-85}{85}$ = $\frac{45}{85}$ =0.53 ×100=53%

Method 2: $\frac{130-85}{85}$ = $\frac{45}{85}$ = $\frac{45}{85}$ ×100%= 45 × $\frac{100}{85}$ %= 45 ×1.2%=53%

When a value or quantity decreases, we can compute percent decrease as in percent increase. Remember that the larger value is original value and comes first in the formula.

Experimental and theoretical probability:

Probability of an event is the chance an event occurs and varies between 0 and 1. In experimental probability, we conduct an experiment, record the number of times an event occurs, measure the number of times the experiment is performed , and then divide the two numbers. Theoretical probability is the number of times an event occurs (favorable outcomes) divided by the number of total outcomes.

P(event)= $\frac{favorable\ outcomes}{number\ of\ total\ outcomes}$

Example 1: Experimental probability

There are 25 marbles in ajar, 12 yellow marbles, 8 red marbles, and 5 blue marbles. Take a marble from the jar, repeat 20 times after returning the marble into the jar. What is the probability of taking a blue marble?

Solution: Let's assume the recorded number of taking blue marble is 5 out of 25 times experimental repetition (25), then

P(blue marble) =$\frac{5}{25}$ =$\frac{1}{5}$

Example 2: Theoretical probability

What is the probability of odd number in a rolling die containing numbers 1-6?

Express the probability in fraction, decimal, ratio, and percent.

Solution: odd numbers are 1, 3, and5. Therefore, the favorable outcome is 3 and the total outcome is 6.

P(odd numbers)=$\frac{3}{6}$ = $\frac{1}{2}$ (fraction)= 50%= 0.5=1:2

Probabilities of independent and dependent events:

Two events are considered independent if the occurrence of event does not affect the likelihood of occurrence of the other event. And two events are dependent if the occurrence of event affects the likelihood of occurrence for the other event.

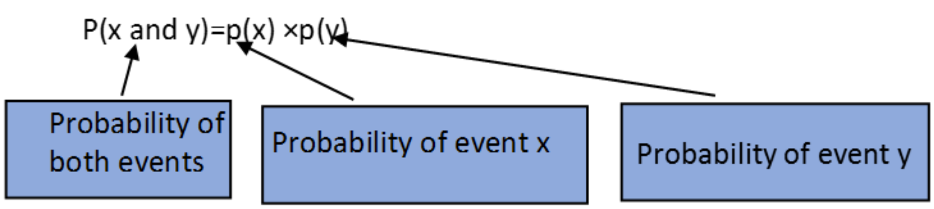
Example:

1. The outcome of flipping a coin does not affect the outcome of flipping other coins. Therefore, the events are independent.

2. Your coach selects a leader for one group of students and then chooses another leader for the second group. Here, the events are dependent. Because there are fewer people available to choose from when the leader of the second group is chosen.

Probability of independent events:

Probability of two independent events, event x and event y, is the product of the probability for event x and the probability for event y.



Finding the probability of two independent events:

1. What is the probability of rolling two 5 when throwing two dices with numbers 1-6?

Dice 1: favorable outcome: 1, total outcomes: 6

Dice 2: favorable outcome: 1, total outcomes: 6

P(two dices rolling 5)=p(dice 1 rolling 5)×p(dice 2 rolling 5)

P(two dices rolling 5)=$\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

2. What is the probability of taking 3 on a deck of 52 cards twice after replacing the card following the first attempt.

First attempt: favorable outcomes: 4, total outcomes: 52

Second attempt: favorable outcomes: 4, total outcomes: 52

P(3 on both attempts)= $\frac{4}{52}$ (probability of 1th attempt) × $\frac{4}{52}$ (probability of 2nd attempt)= $\frac{16}{2704} = \frac{1}{169}$

3. What is the probability of two heads on flipping two coins?

Coin 1: favorable outcome: 1, total outcomes:2, probability:$\frac{1}{2}$

Coin 2: favorable outcome: 1, total outcomes:2, probability:$\frac{1}{2}$

P(two heads)=$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

Probability of two dependent events:

Probability of two dependent events is the product of event 1 and the provability of event 2 after event 1 occurs.

P(event 1 and event 2 after event)=p(event 1) ×p(event 2 after event 1)

Finding the probability of two dependent events:

A jar of marbles contains 3 blue marbles, 5 yellow marbles, and 2 red marbles. What is the probability of taking two blue marbles in two attempts without looking?

Attempt 1: favorable outcome:3 blue marbles, total possible outcomes: 10

marbles altogether, probability: $\frac{3}{10}$.

Let's assume we took a yellow marble in the 1th attempt. Now, we do NOT replace into the jar. This will affect the total number of marbles (total outcomes)

Attempts: favorable outcome:3 blue marbles, total possible outcomes: 9 marbles altogether, probability: $\frac{3}{9}=\frac{1}{3}$

P(two attempts)= p(attempt 1) × p(attempt 2)=$\frac{3}{10} \times \frac{1}{3}=\frac{1}{10}$ 10%

Permutations and combinations:

Permutation is described as different ways to group items together. For instance, writing all possible grouping of 3 letters from a set of 5 letters in the word, let's say BOARD. It is computed using the following formula where "a" represents the number of items which is 5 in the BOARD, and "b" represents the selected grouping size which is 3 as mentioned above, "p" represents permutation, and where order is important as BOA and BAO are two different groupings. Remember that the exclamation sign (!) is used to denote factorial which means to write, for example 5,as 5×4×3×2×1.

a_ p_b=$\frac{a!}{(a-b)!}$

Example: word BOARD:

5_p_3=$\frac{5!}{(5-3)!}=\frac{5!}{2!}=\frac{5\times4\times3\times2\times1}{2\times1}$ =60

Combination is explained as groupings where order of items or letters does NOT matter. In the above-mentioned BOARD example, the three letters grouping BOA is the same as BAO and is written as 5_C_3 in our example. It can be calculated by the following formula.

a_C_b=$\frac{a!}{b!(a-b)!}$ Where a and b represent the same concept as in permutation and C denotes combination or choose.

Example: word BOARD

5_C_3=$\frac{5!}{3!(5-3)!}$ = $\frac{5\times4\times3\times2\times1}{3\times2\times1(2\times1)}$ = 10

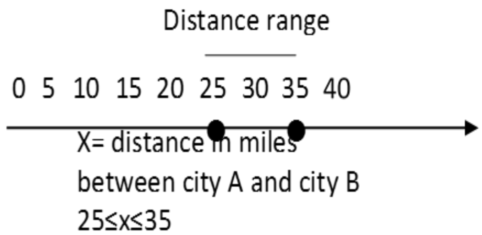
We can also use Pascal's triangle to find number of combination

Chapter 8) Writing and solving inequalities

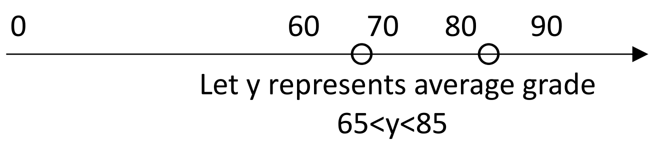
Writing and graphing of one variable inequalities:

In inequalities, we use mathematics symbols >, <, ≥, and ≤ to indicate a range of values (there are multiple possible solutions) instead of specific value (s) as in equations. For example, a>b (a is greater than b), a<b (a is less than b), a≥b (a is greater or equal to b), and a≤b (a is equal or less than b). Flipping the sides in inequality will NOT change the inequality as long as the inequality symbols are reversed, for instance, 2<x<3 is the same as 3>x>2. One way to represent inequality is by using number line. The following examples depicts the number line representation of inequality. An open dot is used to represent < and >, and a closed dot is used to represent≤ and ≥. Open/closed dots are often called "end points."

1. The distance between city A and city B is somewhere between 25-35 miles.



2. The average student grade for mathematics varies among different high schools from around 65 to around 85. The following number line graphs the variation.



One step inequalities:

One-step inequalities are similar to one-step equations EXCEPT for inequality sign. These are inequalities with a variable that has a coefficient other than one and the steps to solve these inequalities involves multiplication, division, addition, and subtraction as in one-step equations. If both sides of an inequality is multiplied or divided by a positive number, the inequality sign remains the same while if the both sides are multiplied or divided by a negative number, the inequality sign must be reversed.

Subtraction/addition rule:

If $x>y$, then $x+b>y+b$

If $x>y$, then $x-b>y-b$

Multiplication/division rule:

If x>y, then xb>yb if b>0

If x>y, then xb<yb if b<0

If x>y, then $\frac{x}{b} > \frac{y}{b}$ if b>0

If x>y, then $\frac{x}{b} < \frac{y}{b}$ if b<0

Example:

1. -13>x-8, -13+8>x-8+8, -5>x
2. -2+y≥5, +2-2+y≥5+2, y≥7
3. a-7≤-15, a-7+7≤-15+7, a≤-8
4. 17+b≤0, -17+17+b≤-17, b≤-17
5. 4+x≤-10, -4+4+x≤-10-4, x≤-14
6. $\frac{a}{5}$<-5, 5 ×$\frac{a}{5}$ < -5 × 5, a<-25
7. -8y≥-72, $\frac{-8y}{-8} \geq \frac{-72}{-8}$, y≤9
8. 0≥9n, $\frac{0}{9} \geq \frac{9n}{9}$, 0≥n
9. -12d<-144, $\frac{-12d}{-12} < \frac{-144}{-12}$, d>12
10. 48≥-12e, $\frac{48}{-12} \geq \frac{-12e}{-12}$, -4 ≤e

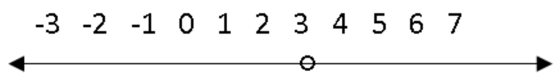
Two-step and multi-step inequalities:

Operations for two-step and multi-step inequalities are very similar to two-step and multi-step equations. The difference is that we MUST flip the inequality symbol when both sides of the inequality is divided or multiplied by a negative number. Here, we will review the operations with several examples.

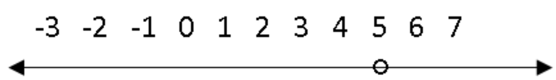
Two-step inequalities:

Example:

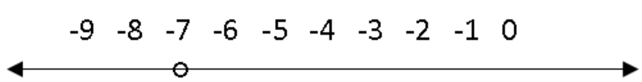
1. 3x-4<5, add +4 to both sides, 3x-4+4<5+4, 3x<9, divide both sides by 3, x<3



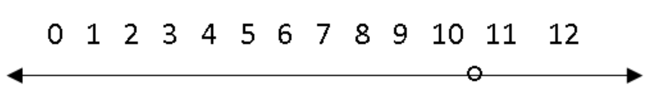
2. 5y+10>25, subtract both sides by -10, 5y+10-10>25-10, 5y>15, divide both sides by 5, y>5.



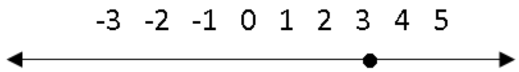
-2a-4<10, add +43. to both sides, -2a-4+4<10+4, -2a<14, divide both sides by -2 and reverse the inequality symbol, a>-7.



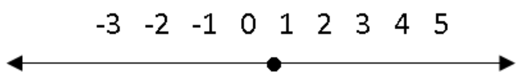
4. $\frac{-2b}{3}$ +5<--2, multiply both sides by -3, (-3) ×($\frac{-2b}{3}$ +5)<--2 ×(-3), reverse the inequality sign 2b-15>6, add +15 to both sides, 2b-15+15>6+15, 2b>21, divide both sides by 2, b>$\frac{21}{2}$



5. $4x-3 \le 9$, add +3 to both sides, $4x-3+3 \le 9+3$, $4x \le 12$, divide both sides by 4, $x \le 3$.



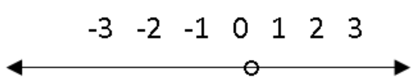
6. $\frac{-5y}{2}+6 \ge 4$, multiply both sides by 2, $2(\frac{-5y}{2}+6) \ge 4 \times 2$, $-5x+12 \ge 8$, subtract both sides by -12, $-5x+12-12 \ge 8-12$, $-5x \ge -4$, divide both sides by -5 and reverse the inequality sign, $x \le \frac{4}{5}$



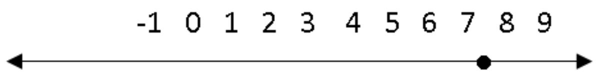
Multi-step inequalities

Example:

1. $5(2a+1)-3<4$, first distribute, $10a+5-3<4$, $10a+2<4$, subtract -2 from both sides, $10a+2-2<4-2$, $10a<2$, divide both sides by 10 and simplify, $a<\frac{2}{10}=\frac{1}{5}$



2. $\frac{3}{2}(4b-2)-3(3x+4) \ge 7$, distribute, $6x-3-9x-12 \ge 7$, $-3x-15 \ge 7$, add +15 to both sides, $-3x-15+15 \ge 7+15$, $-3x \ge 22$, divide both sides by -3 and reverse the inequality sign, $x \le -\frac{22}{3}$



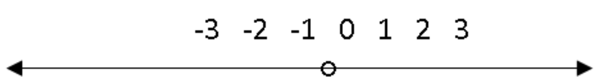
Inequalities with variables on both sides:

Operation steps for solving these inequalities is the same as steps we used to solve equations with variables on both sides of the equal sign. Apply all steps and rules we learned for one-step and two-step inequalities.

Example:

1. 3(2y+1)-2>2(y-1)+1, distribute, 6y+3-2>2y-2+1, add like terms on both sides, 6y+1>2y-1, move variables to one side of the inequality sign and the numbers to other side of the inequality sign. Remember, we MUST change the negative sign to positive and the positive sign to negative as numbers or variables are moved to opposite side.

6y-2y>-2, 4y>-2, divide both sides by 4 and simplify, $\frac{4y}{4} > \frac{-2}{4}$, y>- $\frac{1}{2}$



Compound inequalities

In compound inequalities, we deal with two inequalities that are connected by the word “or” or the word “and”. Variable is constrained by the values of the two inequalities. Individual inequalities must be solved separately and graphed on number line.

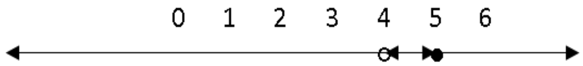
Example:

1. 5<2x-3≤7, 2x-3>5 and 2x-3≤7 should be solved separately:

2x-3>5, 2x-3+3>5+3, 2x>8, x>4

2x-3≤7, 2x-3+3≤7+3, 2x≤10, x≤5

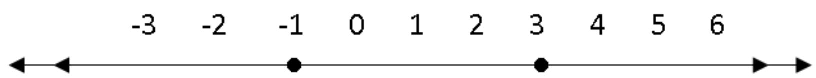
Result: we must graph $4<x\leq5$, intersection of two inequalities.



2. $-2\geq5y+3\geq18$

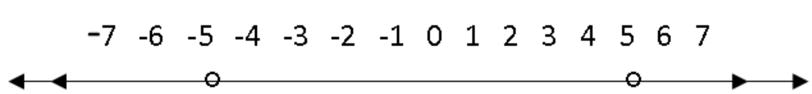
$5y+3\leq-2$, $5y+3-3\leq-2-3$, $5y\leq-5$, $y\leq-1$

$5y+3\geq18$, $5y+3-3\geq18-3$, $5y\geq15$, $y\geq3$



Absolute value equations and inequalities:

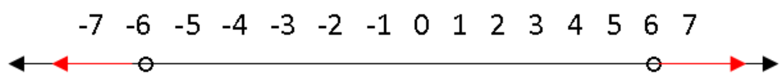
Absolute value of number "b" is shown as IbI and is a distance from 0 on a number line. Absolute value equation is an equation with absolute value expression such as Ix-1I=4 and has two solutions and must be solved for two equations separately, that is, $x-1=4$ and $x-1=-4$. Absolute value can never be negative. Therefore, equations that equal a negative number have no solution. Also, absolute value inequality is an inequality with absolute value expression such as IyI>5 (must be solved for two inequalities) and can be written as compound inequality, $y>5$ and $y<-5$, and solved separately. Let's graph it on a number line.



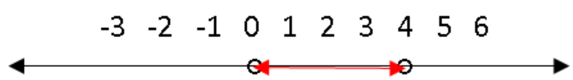
Example:

Absolute value inequality

I2x+1I>6, clear ab1. solute value symbol and identify two inequalities, here with a greater sign and the absolute value of 6, 2x+1>6 or 2x+1<-6.



2. I2y-4I<4, -4<2y-4<4, divide by 2, $\frac{-4}{2} < \frac{2y}{2} - \frac{-4}{2} < \frac{4}{2}$, -2<y-2<2, add 2, -2+2<y-2+2<2+2, 0<y<4.



Chapter 9) Graphs and functions

Displaying data:

After collecting and organizing fata from physical and social studies, the next step is displaying and finally analyzing data. An important step in displaying data is to select a particular graph that best suits your specific type of data for clear and easy understanding of the information. For instance, bar graph, line plot, scatter plot and stem-and-leaf plot are usually used to display numerical fata by looking at the mean, median, count, and the shape (e.g., bell or clustering in scatter plot) of the data.   Also, longitudinal data is best represented by line graph. Highlighting similarity, difference, relationship, or trend will help easy understanding of data. It is also necessary to consider the advantages and disadvantages of different types of graphs including bar graph (comparing data), line graph (change versus time), pictograph (frequency and compare of data), line plot (organizing one group of data), histogram (frequency and comparing data), table, chart, scatter plot (testing for correlation of two data sets and calculating the line and the curve of best fit), stem-and-leaf plot (showing frequency, data is grouped based on place value, using digit in the greatest place), box-and-whisker plot (range of values median, quartile, outlier), and pie graph (parts or percentages of a whole) for display.

Intro to the coordinate plane and relations:

Starting with Rene Descartes, a French mathematician and philosopher, a new era of creating a bridge the word of algebra and geometry. Rene was able to visualize an equation with, for instance, x and y variables in y=x-2, by giving values to x and finding the corresponding y values in the equation. Conventionally, these x and y values are called **Cartesian coordinates** as depicted on two-dimensional Euclidian coordinate plane with horizontal (x axis) and vertical (y axis) axes. Then, by drawing the x and the y coordinates, generating a point of intersection for x and y coordinates, and connecting the point of intersection for several pairs of x and y coordinates represented as (x, y). this way, we can visualize the algebraic equations with different shapes.

Example:

y=x-2

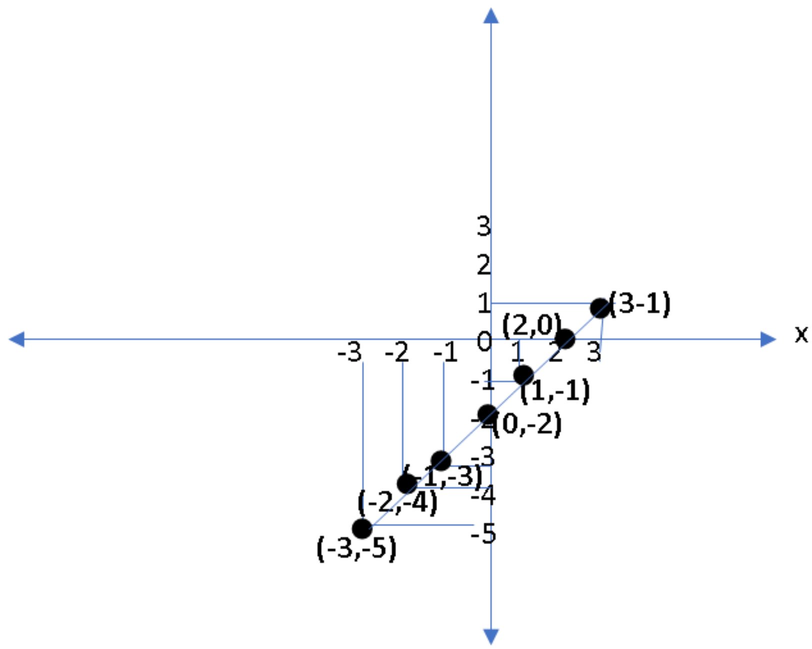
Step 1: give x values and find corresponding y values from equation.

This is a "relation" organized in a table below and it is a function relation since every x input generate a single defined y output. In other word, in function relation, every member of the domain is associated or mapped to a specific member of the range. The set of x-coordinates (generally inputs for a function) in the relation is that create defined members of a range called domain and the set of y-coordinates (function output) in the relation is called range.

X	y
-3	-5
-2	-4
-1	-3
0	-2
1	-1
2	0
3	1

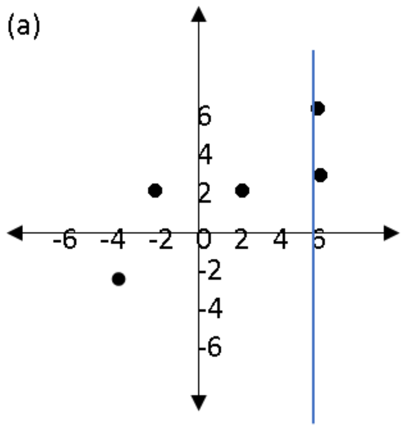
Step 2: Find the intersection for (x, y) values or coordinates on the coordinate

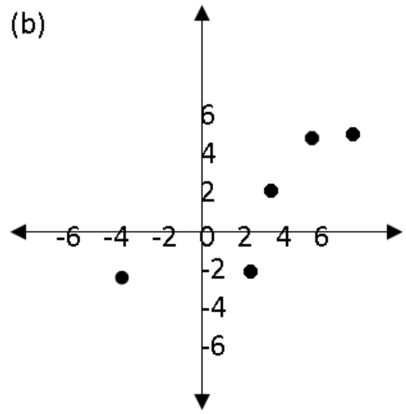
plane and connect the dots. This is an example of a linear equation (straight line) where the rate of change in y over the rate of change in x is constant.

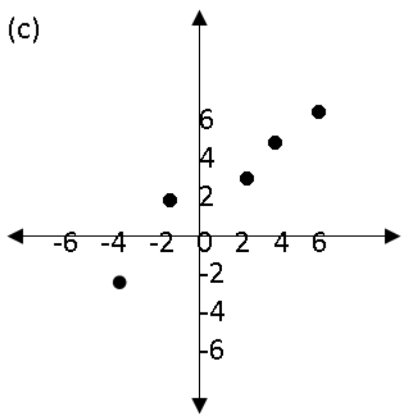


Identifying functions:

A function is a set of ordered pairs or a relation in which every member of the domain (input or x-coordinate) is only associated or mapped to one and only one member of the range (output or y-coordinate). No repeated x value is allowed, however, repeated y value is permitted. A set of ordered pairs can most usually specified in a list or a table, in an equation that generates the ordered pairs, and in a graph that represents the ordered pairs. The graph can show distinct ordered pairs or all the ordered pairs on a line or curve and a vertical line test is used to test whether thee ordered pairs represent function on a graph. For instance, in the following graph relations, there is no vertical line that can pass through two ordered pairs on the graph relations “b” and “c”, but there is a vertical line that passes through two ordered pairs, (5, 3) and (5, 6) on graph relation “a”. Therefore, ordered pairs on graph relations “b” and “c” represent function, but graph relation “a” does NOT represent the graph of a function.







Example:

Is the relation given by set of ordered pairs below, a function?

a) {(-2,1), (-1,1), (0,2), (1,3), (2,3)}

b) {(--4,-2), (-2,3), (-2,7), (3,8)}

Solution:

Step 1. Write the domain and the range

part a:

Domain: {-2, -1, 0, 1, 2}

Range: {1, 1, 2, 3, 3}

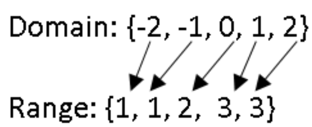
Part b:

Domain: {-4, -2, -2, 3}

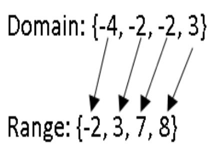
Range: {-2, 3, 7, 8}

Step 2. Draw association: part “a” represents a function since no element of the domain is associated with two different elements of the range. Part “b” does NOT represent a function because one element of the domain (-2) is associated with two different elements of the range (3 and 7).

Part a:



b:



Which of the following equations represent a function?

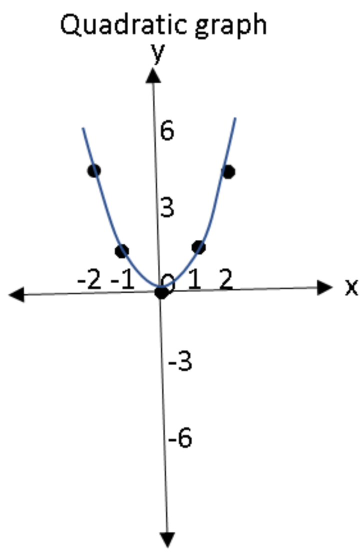
a) $3x^2+y^2=5$; solve for y as an output, $y^2=5-3x^2$, $y=\pm\sqrt{(5-3x^2)}$, it is not a function because there are two outputs (y) for every input (x).

b) $7x+y=9$, $y=9-7x$, represents a function.

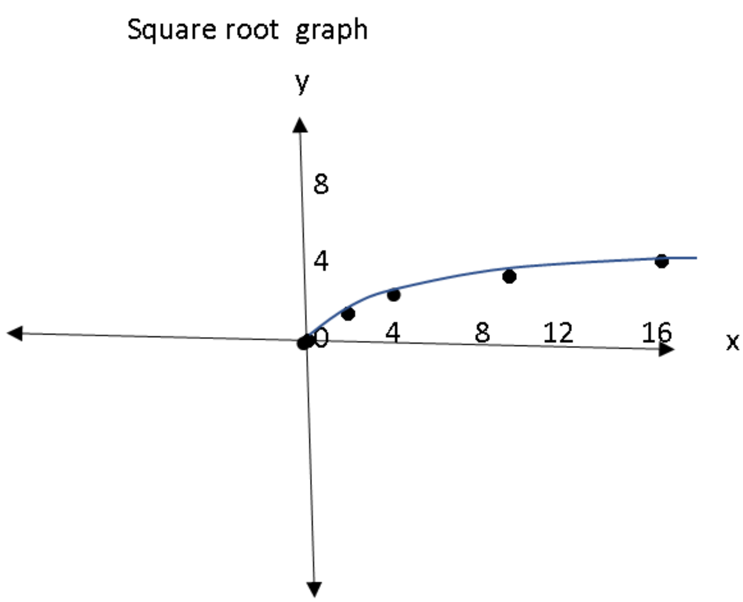
Function rules, tables and graphs:

There are three ways to represent function, function rule, function table, and graph. A unction table is an organized presentation of of inputs (independent variable or x value), function rule, and output (dependent variable or y value). Function table is used to graph a function. In the following examples, function table with independent variable (input or x value), rule (f(x)),and dependent variable (output or y value) as well as corresponding different graph types are presented. We will discuss in detail each graph type later chapters.

x	$f(x)=x^2$	y	(x,y)
-2	4	4	(-2,4)
-1	1	1	(-1,1)
0	0	0	(0,0)
1	1	1	(1,1)
2	4	4	(2,4)

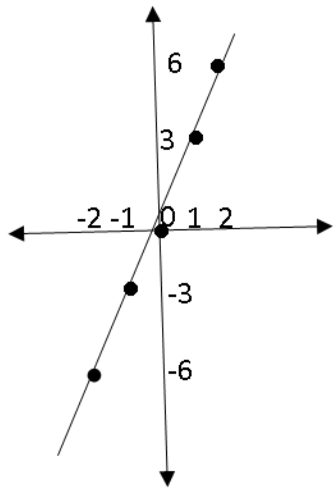


x	$f(x)=\sqrt{x}$	y	(x,y)
16	4	4	(16,4)
9	3	3	(9,3)
4	2	2	(4,2)
1	1	1	(1,1)
0	0	0	(0,0)



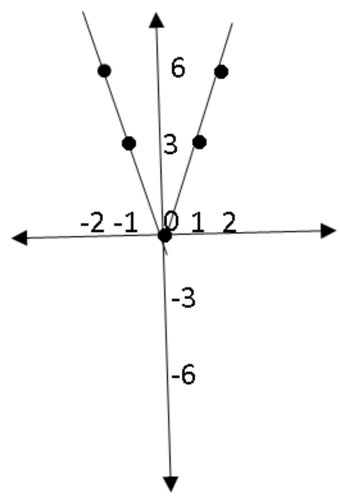
x	f(x)=3x	y	(x,y)
-2	3(-2)	-6	(-2,-6)
-1	3(-1)	-3	(-1,-3)
0	3(0)	0	(0,0)
1	3(1)	3	(1,3)
2	3(2)	6	(2,6)

Linear graph



x	f(x)=3lxl	y	(x,y)
-2	3(2)	6	(-2,6)
-1	3(1)	1	(-1,3)
0	3(0)	0	(0,0)
1	3(1)	3	(1,3)
2	3(2)	6	(2,6)

Absolute value graph



Writing a function rule:

To find the function rule from a function table or in real world situation, we to look for a pattern that models relationship between independent variable (x values) and dependent variable (y values). The first step is to find whether the function is linear or higher degree of function. We can start with looking at the rate of change for x and y values in the function table. If the first rate of change is for y values over the rate of changes of y values is constant, then the function is linear. If the second rate of changes for y values over the rate of changes for x values is constant, then the function is quadratic.

Example:

1.

x	y
1	2
2	4
3	6
4	8

The 1th rate of changes for y values over the rate of changes for x values is 2. Therefore, the function is linear with a slope of 2. We can find the function rule using the general formula for linear equation, $y=mx+b$, to find the y intercept (b value) by substituting one of the points in the function table as follows:

$y=mx+b$, $2=2(1)+b$, $b=0$, and the function rule is $y=2x$

2.

x	y
1	2
2	5
3	10
4	17

Here, the 1th rate of change for y values is not constant but the second rate of changes for y values is constant at 2. Therefore, the function rule is quadratic. By looking at the pattern of changes in y values we can write the function rule as $y=x^2+1$

3.

x	y
-2	4
-1	2
0	0
1	2
2	4

The pattern of change in y values, -2, -2, 0, 2, 2, is typical pattern for V-shape of absolute value function rule with power one. The function rule is written as $y=2|x|$.

Arithmetic sequences:

A sequence is a list of numbers with specific pattern where each number is a tern for the sequence. There are different types of sequences with different patterns.

{1,2,3,4,5,6,7,8,9,…}

In a sequence that each term is generated by adding or subtracting a constant number is called arithmetic sequence. In our above example, each term is made by adding 1 to previous term a d is an arithmetic sequence The terms in arithmetic sequence is identified by recursive or explicit formulas. Recursive formula is written as

$a_n=a_{(n-1)}+1$ and the explicit formula is explained as

$a_n=a_1+(n-1)d$ where d is the difference of two consecutive terms, a1 is the 1thterm in the sequence, and a_n is the nth term in the sequence.

Chapter 10) Graphing equations

Slope:

Slope of a line is indicative of how fast the line changes. It is a measure of rise or the rate of change in the y axis over the run or the rate of change in the x axis in a coordinate plane for a line. It can be used for straight line, a line tangent to a curve, or a curve.

Slope of a linear equation, y=mx+b:

Slope of a linear equation could be positive (line is changing upward) or negative (line is changing downward). One way to find the slope is when we have the equation for the line and meet the following conditions:

1. Only two variables with no exponent

2. The equation can be simplified to the form y=mx+b, where m and b are constants such as 2,6,or -3. Then, m is the slope and b is the y intercept

Example 1:

5x-y=7, isolate variables and the equation to the form y=mx+b

Y=5x-7, slope or m=5

Example 2:

4x+6=2y-4, 2y=4x+6+4, 2y=4x+10, divide both sides by 2, y=2x+5, m=2

Finding slope with two points, $m=(y_2-y_1)/(x_2-x_1)$:

We can also find the slope using two points in a coordinate plane and putting in the point-slope formula, $m=(y_2-y_1)/(x_2-x_1)$. Remember that the following:

a. Horizontal line has a zero slope

b. Vertical line has undefined slope

c. Larger slopes are steeper lines and smaller slopes are more gradual

d. Positive slopes go higher and higher to the right and negative slopes go lower and lower to the right

Example:

Find the slope of a line passing through two coordinate points, (5,2) and (7,6).

$m=(y_2-y_1)/(x_2-x_1)$, m=(6-2)/(7-5)=4/2=2

Finding slope of a curve using derivative:

We will discuss this section later after learning how to do derivatives, but we mention one example of quadratic equation. Using derivative will give us the slope or the rate of change at a single point on line or s curve instead of the entire line.

Example:

Find slope of equation, $3x^2+5x$ at coordinate point (3,4).

$f'(x)= 6x+5$, plug in the x coordinate at point (3,4), $f'(x)= 6(3)+5$, $f'(x)=23$, the slope of the $f(x)=3x^2+5x$ at the point (3,4) is 23.

Using x- and y- intercept:

x-intercept is where the graph of an equation crosses the x-axis and, therefore, $y=0$. And y-intercept is where the graph of an equation crosses the y-axis and, therefore, $x=0$.

Example: find the x and y intercepts of equation, $y=x^2-16$

x-intercept where $y=0$, $x^2-16=0$, $x=\pm 4$ are the x-intercepts

y-intercept where $x=0$, $y=0-16$, $y=-16$ is the y-intercept

Function forms:

Linear function forms:

When f(x) is linear, then the graph for y=f(x) is straight line. Linear function is written in three different ways: 1. Point-slope form $y-y_1=m(x-x_1)$, 2. Slope-intercept form, $y=mx+b$, and 3. General or standard form, $Ax+By=C$. The parameter m ($\frac{\Delta y}{\Delta x}$) in the slope-intercept and point-slope forms is the slope of the line. In the slope-intercept form, the parameter b is y-intercept where the x value is 0. The point (x_1,y_1) in the point-slope form is a point on the line. In the general form, $m=-A/B$ when $B\neq 0$.

Example: A line passes through the points (-2,12) and (4,0). Write the equation of the line in point-slope, slope-intercept, and the general forms.

1. The first step is to find the slope.

Slope=m=$\frac{\Delta y}{\Delta x}$ =$\frac{0-12}{4-(-2)}$=$\frac{-12}{6}$=-2

2. Write the point-slope form by plugging in the m value and one of the given points

$y-y_1=m(x-x_1)$, **y-12=-2(x+2)** is the point-slope form

3. Manipulate the point-slope form to find the slope-intercept form

y-12=-2(x+2), distribute -2, y-12=-2x-4, add 12 to both sides, y-12+12=-2x-4+12, simplify, **y=-2x+8** is the slope-intercept form

4. Manipulate the slope-intercept form to find the standard form

y=-2x+8, add 2x to both sides, y+2x=-2x+8+2x, **y+2x=8** is the standard form

Quadratic function forms

Quadratic function can be expressed in:

1. factored form, f(x)=a(x-p)(x-q)

2. vertex form, $f(x)=a(x-h)^2+k$ where the point (h,k) is the vertex of the parabola and x=h is the axis of symmetry

3. standard form, $f(x)=ax^2+bx+c$ where the coordinates for the vertex point (h,k) can be computed using h=$\frac{-b}{2a}$ and k=f(h)

Example: $y=x^2-4x+7$ is a standard form of quadratic function. Find the vertex form for this function.

Step 1. Isolate 1th degree and the 2nd degree variable form constant, $\mathbf{x^2-4x=y-7}$, Step 2. Add 4 to both sides, $\mathbf{x^2-4x+4=y-7+4}$,

Step 3. Find a perfect square by figuring a number that is equal to 4 when squared and is equal to -4 when multiplied by 2, $\mathbf{(x-2)^2=y-3}$

Step 4. Rearrange to vertex form, $\mathbf{y=(x-2)^2+3}$

Writing equation of a line:

Finding a line equation with a given point (x_1,y_1) on the line and the line slope:

The strategy is to plug in the known information, point (x_1,y_1) and slope, in the point-slope form, $y-y_1=m(x-x_1)$, and then convert it slope-intercept or standard form form as needed.

Example: Find equation of a line that passes through point (3,6) with a slope of 2.

Step 1. Plug in the numerical values in the point-slope form, $y-y_1=m(x-x_1)$,

y-5=2(x-3)

step 2. Convert to slope-intercept form, y=mx+b

Distribute 2 in y-5=2(x-3) point-slope form y-5=2x-6

step 3. Add 5 to both sides of the equal sign and rearrange to slope-intercept form

y-5+5=2x-6+5, y=2x—1

Finding line equation with two known points on the line:

When given two points on the line, we use the slope formula, $m=(y_2-y_1)/(x_2-x_1)$, to find the slope value.

Example: What is the equation of a line passing through points (-3,5) and (4,-3).

Ste- 1. Find slope

m=(-3-5)/(4-(-3)), m=(-8)/(7)=$\frac{-8}{7}$

step 2. Plug in one of the points and the slope value in the point-slope form to find the equation.

$y-y_1=m(x-x_1)$, y-5= --$\frac{8}{7}$ (x+3)

step 3. Convert to the form desired, let's convert to slope-intercept form

1. Distribute --$\frac{8}{7}$, y-5=--$\frac{8}{7}\ x-\frac{24}{7}$
2. Add 5 to both sides, y-5+5= - -$\frac{8}{7}\ x-\frac{24}{7}$ +5,
3. Simplify, **y=- -**$\mathbf{\frac{8}{7}\ x+\frac{11}{7}}$

.

Parallel and perpendicular lines:

Parallel lines have the same slope and perpendicular limes have negative reciprocal slopes. Using this information, we find the equation of a line perpendicular or parallel to given line.

Example: Write the equation of a line perpendicular to line 4x+y=3a and passes through the point (-2,-5) and the equation of a second line that passes through the point (3,4) and is parallel to the same given line

Step 1. The given equation is in standard form. It is easily visible if we transform the given equation to slope-intercept form, y=mx+b where m is the slope.4x+Y=3, subtract -4 from both sides, 4x+Y-4x=3-4x, **y=-4x+3**

Step 2. From step 1, we see that the slope of the given equation is m=-4 .

Therefore, a line perpendicular to this line is m=$\frac{1}{4}$ and a second line parallel to the given line is m=-4

Step 3. The easiest way is to plug in the coordinate values and the slope for each line for point-slope form. Then, transform to desired equation form as described above.

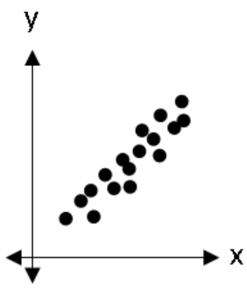
Perpendicular line: y-(-5)=$\frac{1}{4}$ (x+2), distribute $\frac{1}{4}$, y+5=$\frac{1}{4}x+\frac{1}{2}$, subtract -5 from both sides, y=$\frac{1}{4}$ x- $\frac{9}{2}$

Parallel line: y-4=-4(x-3), distribute, y-4=-4x+12, add 4 to both sides, y-4+4=-4x+4, y=-4x+16

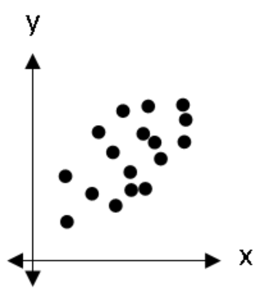
Scatter plots and correlations:

Scatterplot is similar to line graph in that they are graphed on horizontal (x-axis) and vertical (y-axis) axes. However, scatterplot is more specifically used to display how two variables are related. In other words, scatterplot shows the relationship or association of independent and dependent variables. This relationship is called correlation. In scatterplot, individual data points are displayed as scattered points with a patter or trend in the relationship of two indicated variables. The pattern is best visualized by drawing a line of best fit or trend line where the line models the trend for prediction based on past information. The closer t he data points of scatterplot is in the pattern, the stronger the relationship. There are three different types of correlations, positive correlation, negative correlation, and no correlation. In positive correlation, when independent variable increases on the x-axis, the dependent variable will also increase on the y-axis and the scatterplot pattern is displayed from lower left to upper right in the first quadrant of the coordinate plane (there is no need for negative x or y values). If dependent variable (y value) decreases after independent variable (x value) increases, then the correlation is negative correlation and the scatterplot trend is graphed from upper left to lower right of the first quadrant in the coordinate plane. When scatterplot of data points shows no pattern in the graph, there is no correlation between x and y variables. Correlation is more precisely described as what is called **coefficient of correlation** that varies within -1 to 0 to +1 where -1 is perfect negative correlation, 0 is no correlation, and +1 is perfect positive correlation. Absolute value of the coefficient of correlation shows the strength of correlation.

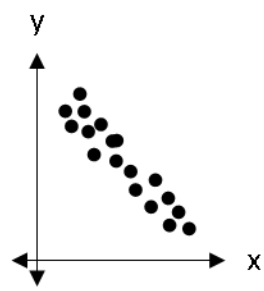
(a). Positive correlation with tightly clustered data points and high positive correlation of x and y values



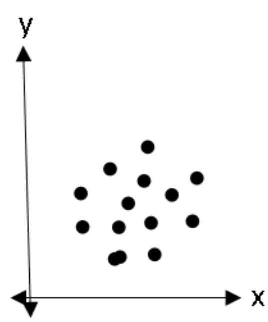
(b). Positive correlation with loosely clustered data points and low correlation of x and y values



(c). Negative correlation with tightly clustered data points and high negative correlation of x and y values



(d).No correlation with no pattern in data points and zero correlation of x and y values



Chapter 11) Solving systems of equality and inequality

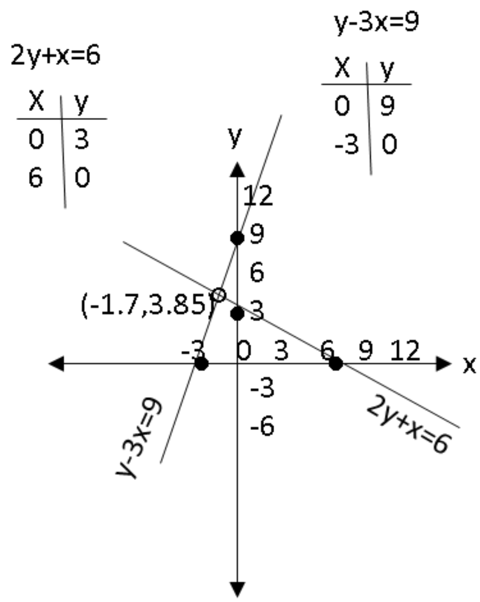
Solving systems by graphing:

Solving system of linear equations by graphing:

The solution to a system of equations is the point of intersection of two graph and must satisfy both equations.

Example: solve graphically the system of equations, 2y+x=6 and y-3x=9

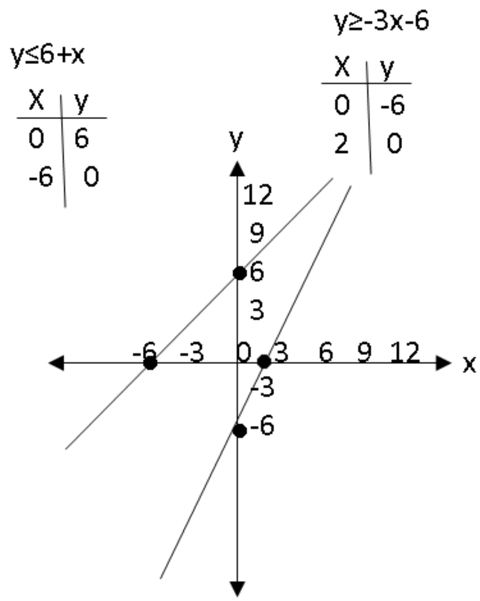
One way to graph each equation is find two points and connect the dots. The intersection of individual graphs (-1.7,3.85) is the solution to the system and satisfies both equations.



Solving system of inequalities by graphing:

Solution of a system of inequalities is ordered that satisfies all inequalities in the system. We first graph individual inequalities and shade the overlapping area that satisfies all inequalities which is the solution to the system. The boundary of inequalities is the equation of the lines. All the points between the boundary lines in the graph below satisfy both inequalities and is the solution to the system of inequalities.

Example: Solve the system of inequalities, y≤6+x and y≥3x-6.



Solving systems using substitution:

Solving linear equation system:

We solve for one of the variables in the one of the equations and, then plug it into the other equation. Following example will explain the process.

Example: Solve the system of equations, x-2y=6 and 4y+5x=10, by substitution.

Step 1. Let's solve for variable x in equation x-2y=6.

Add 2y to both sides, x-2y+2y=6+2y, then x=6+2y

Step 2. Substitute into the other equation.

4y+5 (6+2y) =10, distribute 5, 4y+30+10y=10, 14y+30=10, subtract 30 from both sides, 14y+30-30=10-30, 14y=-20, divide both sides by 14 and simplify, y= $-\frac{20}{14}$ $=-\frac{10}{7}$

Step 3. Plug the y value into one of the equation and solve for x value.

x-2($-\frac{10}{7}$)=6, x+$\frac{20}{7}$ =6, subtract $\frac{20}{7}$ from both sides, x+$\frac{20}{7}$ - $\frac{20}{7}$ =6 - $\frac{20}{7}$, x=$\frac{22}{7}$

Step 4. The point ($\frac{22}{7}$, - $\frac{10}{7}$) is the solution to the system of equations.

Solving systems using elimination:

In this method, we have to create variables with the same coefficient using multiplication and division, then We use subtraction and addition to solve a system of linear equalities through elimination.

Example: Solve the system of equations,2y-3x=6 and y+2x=4, by elimination.

Step 1. Create the same coefficient for one of the variables in both equations. Let's choose the x variable.

2y-3x=6, multiply both sides by 2, 4y-6x=12

y+2x=4, multiply both sides by 3, 3y+6x=12

step 2. After creation of the same coefficient for x variable, now we add the equations to eliminate x variable and find the value for the other variable.

4y-6x+3y+6x =12+12. 7y=24, y=$\frac{24}{7}$

Step 3. Plug the y value into of the equations to find the x value.

$\frac{24}{7}$+2x=4, subtract $\frac{24}{7}$ from both sides, $\frac{24}{7} - \frac{24}{7-}$ +2x=4- $\frac{24}{7}$, 2x=$\frac{4}{7}$, divide both sides by 2 and simplify, x=,$\frac{2}{7}$

Step 4. The coordinate point ($\frac{2}{7}$,$\frac{24}{7}$) is the solution to the system and satisfies both equations.

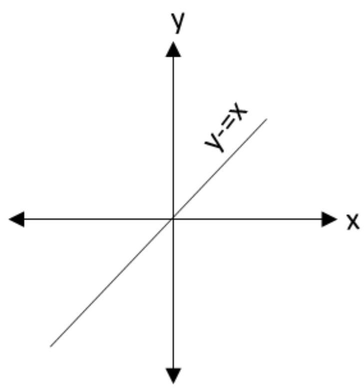
Parent functions:

Parent function is the simplest function of all family of functions. Below, seven parent functions.

1. First parent function is y=x which is a line passing through the origin with a slope +1. Domain and range for this is:

D=all real numbers

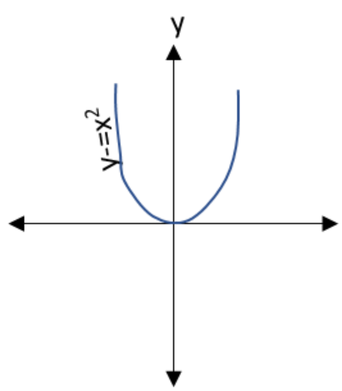
R=all real numbers



2. $Y=x^2$, a parabola that opens up with vertex at(0,0)

D: all real numbers

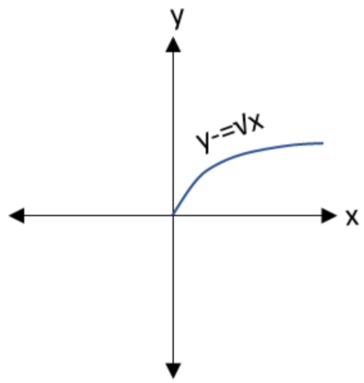
R: y≥0



3. $Y=\sqrt{x}$

D: x≥0

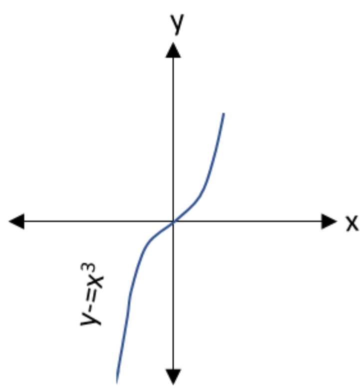
R: y≥0



4. $Y=x^3$

D: all real numbers

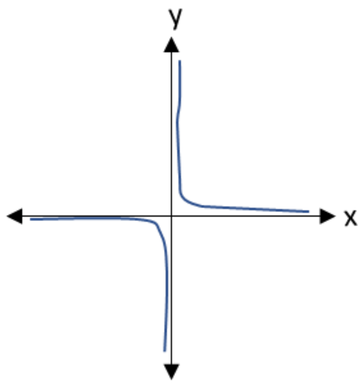
R: all real numbers



5. $\frac{1}{x}$

D: all real numbers, x≠0

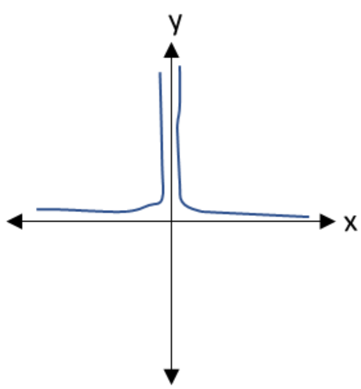
R: all real numbers, y≠0



6. $1/x^2$

D: all real numbers, $x \neq 0$

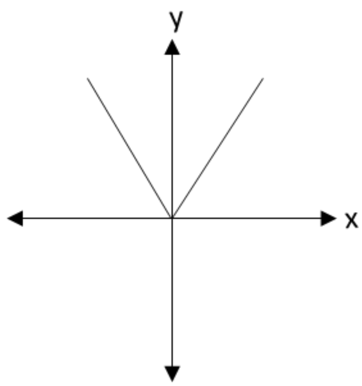
R:$y>0$



7. Y=IxI , is a V-shape curve as shown below.

D: all real numbers

R: $y \geq 0$



Chapter 12) Exponents and polynomials

Exponents:

Exponents tell us how many times a variable or a particular number should be multiplied. The following rules must be remembered when working with variables or numbers that are raised to a power.

1. Multiplication of powers. Add the exponents to get the product of two powers.

$x^3 . x^b = x^{3+b}$, Example: $y^2 . y = y^3$

2. Division of powers. Subtract the exponents to divide two powers.

$x^a / x^b = x^{a-b}$, $x \neq 0$, Example: $x^5 / x^2 = x^{4-2} = x^2$

3. Raise a power to a power. Just multiply the exponents.

$(x^a)^b = x^{a \times b}$, Example: $(x^3)^2 = x^{3\times2} = x^6$

4. Power of a product is the product of each factor to the power.

$(xy)^a = x^a y^a$, Example: $(xy)^4 = x^4 y^4$,

5. Raising a quotient to a power is the raising of the numerator and denominator to the power.

$(x/y)^a=x^a/y^a$, $y\neq0$, Example: $(x/y)^3=x^3/y^3$,

6. Variable or a number to the power zero is 1.

$x^a/x^a=x^{a-a}=x^0=1$, $x\neq0$,

7. Negative exponents are reciprocals of positive exponents.

$x^{-a}=1/x^a$, $x\neq0$, also $x^a=1/x^{-a}$, $x\neq0$, Example: $x^{-2}=1/x^2$,

8. Nth root of a variable x^m is x to the power of m/n.

$\sqrt[n]{x} = x^{1/n}$, Example: $\sqrt[4]{x} = x^{1/4}$

Polynomial operations:

Polynomials are composed of multiple terms containing coefficient multiplied by a variable with non-negative exponent and constant.

Addition of polynomials:

Like terms with the same variable exponent are combined,

Example 1: add polynomials $5x^2+3x-5$ and $2x^2+4x+3$

Step 1. Align the polynomials vertically

$5x^2+3x-5$

$+$

$2x^2+4x+3$

Step 2. Add like terms

$5x^2+5x^2$ =**$8\ x^2$**, 3x+4x =**7x**, -5+3=**-2**,

Solution: $8x^2$ +7x-2

Example 2: What is the solution of polynomials $5x^3+6x$ and $3x^2+2x-1$.

$5x^3+6x$

+

$3x^2+2x-1$

Solution: $5x^3+3x^2+(5x+2x)-1=$**$5x^3+3x^2+5x-1$**

Subtraction of polynomials:

When subtracting polynomials, pay special attention to positive and negative signs of the polynomial terms before and after subtraction.

Example: Subtract polynomial $7y^2+4y-3$ from$9y^2-2y+4$.

Step 1. Align vertically and apply sign rule for terms within parenthesis (negative times negative is positive, negative times positive is negative, and positive times positive is positive).

$9y^2-2y+4$

$-(7y^2+4y-3\ )= -7y^2-4y+3$

Step 2. After apply sign rule, combine the two polynomials.

$9y^2-2y+4$

+

$-7y^2-4y+3$

Step 3. Continue as addition

Solution: $(9y^2 -7y^2) +(-2y-4y)+(4+3)=$**$2y^2$-6y+7**

Multiplication of polynomials:

Each tern of one polynomial must be multiplied by all terms of the second polynomial.

Example:

1. x(x2+x-1) $=x^3+x^2-x$
2. $-2(-x^2+x-5)=2x^2-2x+10$
3. $(x+1)(x^2-2x+3)$

Step 1. Here we have more complicated problem. Multiply x and 1 terms in the first parenthesis by $(+x^2)$, $(-2x)$, and $(+3)$ in the second parenthesis.

$(x)(x^2)+(x)(-2x)+(x)(3)+(1)(x^2)+(1)(-2x)+(1)(3)=x^3-2x^2+3x+x^2-2x+3$

Step 2. Combine like terms and solve the multiplication.

x^3-x^2+x+3

4. $(2x^2+3x-2)(3x^2-5x-4)$

Step 1. Here we three terms in each polynomial. We need to multiply each term in the first polynomial containing terms, $+2x^2$, $+3x$ and -2 by all terms in the second polynomial.

$(2x^2)(3x^2)+(2x^2)(-5x)+(2x^2)(-4)+(3x)(3x^2)+(3x)(-5x)+(3x)(-4)+(-2)(3x^2)+(-2)(-5x)+(-2)(-4)$

Step 2. Multiply and combine like terms.

$6x^4 -10x^3 -8x^2 +9x^3 -15x^2 -12x-6x^2 +10x+8= 6x^4 -x^3 -29x^2 -2x+8$

Division of polynomials:

The best way is to explain the process is during an example.

Example: Divide l polynomial $4x^3-2x^2+6x-8$ by x-2

Step 1. Set up the division

$$x-2\overline{)\,4x^3-2x^2+6x-8}$$

Step 2. Divide the term with the highest degree ($4x^3$) in the polynomial $4x^3-2x^2+6x-8$, multiply by x-2, subtract from polynomial, and write the quotient ass below.

$$\begin{array}{r} 4x^2 \\ x-2\overline{)\,4x^3-2x^2+6x-8} \\ -(4x^3-8x^2) \\ \hline 6x^2+6x-8 \end{array}$$

Step 3. The result of subtraction or remainder, 6x2+6x-8, is used as new polynomial and divided by x-2, and the process is continued until the highest variable degree in the remainder is less than the degree in the divisor (x-2) which is a constant for this example.

$$
\begin{array}{r l}
 & 4x^2+6x+18 \\
x-2 \big) & 4x^3-2x^2+6x-8 \\
 & -(4x^3-8x^2) \\
\hline
 & 6x^2+6x-8 \\
 & -(6x^2-12x) \\
\hline
 & 18x-8 \\
 & -(18x-36) \\
\hline
 & 28
\end{array}
$$

Divide the variable with the highest degree ($4x^3$) in $4x^3$-$2x^2$+6x-8 by the x variable in divisor(x-2)

Multiply by divisor and subtract as shown

Continue the process until the highestt degree in remainder polynomial is less than the divisor degree

Step 4. Solution is written as:

($4x^3$-$2x^2$+6x) / (x-2)= $4x^2$+6x+18+ [28/($4x^3$-$2x^2$+6x)]

Binomial operations:

Binomials are polynomials with two terms. The addition, subtraction, multiplication, and division operations are similar to polynomial operations and much easier. Review part B of this chapter.

Chapter 13) Polynomials and Factoring

Introduction to factoring:

Composite numbers are numbers that can be written as a product of two numbers other than 1 and itself such as 16 and 18. Numbers 16 and 18 are the products of 1×16, 2×8,4×4 and 1×18, 2×9, 3×6, respectively. Positive integers 1, 2, 4, 8, and 16 and 1,, 2, 3, 6, 9, and 18 are factors for the numbers 16 and 18, respectively. Factoring is the process of writing factors of a composite number or composite expression is found. For example, -1, -2, -3, -4, -6, -1, 2, 3, 4, 6, and all factors of positive integer 12 because it number 12 can be written as products of -1×-12, -2×-6, -3×-4, 1×12, 2×6, and3×4. Similarly, the factors for polynomial expression x^2+7x+12 is (2x+4)(2x+3).

Factoring:

Factoring is important for simplifying expressions, fractions, and solving equations.

Finding greatest common factor (GCF):

Greatest common factor (GCF) or divisor (GCD) for two or more integers is the largest positive number that divides into two or more integers. For example, the greatest common factor of 8 and 12 is 4. To compute the GCF for 48 and 12, we list all possible factors for integers, numbers that divide integers without leaving remainder, and then choose the largest factor common for all integers as underlined below.

Factors for 12={1, 2, 3,4, 6, <u>12</u>}

Factors for 48={1, 2, 3, 4, 6, 8, <u>12</u>, 16, 24}

Also, GCF for two or more algebraic terms is determined by finding the GCF for coefficient (as above) times the GCF for the variable (s). When finding the GCF for variable, choose the variable with the lowest exponent.

Example: Find the GCF for the following terms.

$2xy$ and $4x^2y^2$

GCF for coefficients 2 and 4=2

GCF for variables xy and $x^2y^2=xy$

GCF for two terms $2xy$ and $4x^2y^2=2xy$

1. $9a^2b^3$ and $27a^3b^2$

GCF for the coefficients 9 and 27=9

GCF for the variables=a^2b^2

GCF for the terms=$9a^2b^2$

2. $14x^2y^3z^4$, $21x^4y^3z^2$, and $7xy^2z$

GCF for the coefficients=7

GCF for the variables=xy^2z

GCF for the terms=$7zy^2z$

Factoring special products:

Knowing the following special or common products is necessary in algebra, particularly when you are preparing for standardized tests.

Distributive law: $a(b+c)=ab+ac$

Example 1: $5x(2y-3)=10xy-15x$

Example 2: $-2y(3x-y)==-6yx+2y^2$

Difference of two squares: $(a-b)(a+b)=a^2-b^2$

Example 1: $(2x-3y)(2x+3y)=4x^2-9y^2$

Example 2: $(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})=x-y$

Square of a sum: $(a+b)^2=a^2+2ab+b^2$

Example 1: $(4x+3y)2=(4x)^2+2(4x)(3y)+(3y)^2=16x^2+24xy+9y^2$

Example 2: $(2x+3\sqrt{y})^2=(2x)^2+2(2x)(3\sqrt{y})+(3\sqrt{y})^2=4x^2+12x\sqrt{y}+9y$

Square of a Difference: $(a-b)^2=a^2-2ab+b^2$

Example 1: $(2x-y)2=(2y)^2-2(2ax)(y)+y^2=4x^2-4xy+y^2$

Example 2: $(2x^2y-xy^2)^2=(2x^2y)^2-2(2x^2y)(xy^2)+(xy^2)^2=4x^4y^2-4x^3y^3+x^2y^4$

Cube of a sum: $(a+b)^3=a^3+3x^2y+3xy^2+y^3$

Example 1. $(2x^3+3y^2)^3=(2x^3)^3+3(2x^3)^2(3y^2)+3((2x^3)(3y^2)^2+(3y^2)^3=8x^9+18x^6y^2+18x^3y^4+27y^6$

Example 2. $(2x+y)^3=(2x)^3+3(2x)^2(y)+3(2x)(y)^2+(y)^3=8x^3+6x^2y+6xy^2+y^3$

14C6. Cube of a difference: $(a-b)^3=a^3-3x^2y+3xy^2-y^3$

Example 1. $(2x-3y)^3=(2x)^3-3(2x)^2(3y)+3(2x)(3y)^2-(3y)^3=8x^3-18x^2y+18xy^2-27y^3$

Example 2. $(3x-y)^3=(3x)^3-3(3x)^2y)+3(3x)(y)^2-(y)^3=27x^3-9x^2y+9xy^2-y^3$

Sum of two cubes: $a^3+b^3=(a+b)(a^2-ab+b^2)$

Example 1. $8x^3+y^3=(2x+y)(4x^2-2xy+y^2)$

Example 2. $x^3+27y^3=(x+3y)(x^2-3xy+9y^2)$

Difference of two cubes: $a^3-b^3=(a-b)(a^2+ab+b^2)$

Example 1. $8x^3-27y^3=(2x-3y)(4x^2+6xy+9y^2)$

Example 2. $x^3-y^3=(x-y)(x^2+xy+y^2)$

Chapter 14) Quadratic equations/functions

Graphs of quadratic functions:

Quadratic equations are equations with the highest degree of square (second degree). The graph of a quadratic function looks like a letter "U". It is generally written in standard form as $y=ax^2+bx+c$ where a, b, and c are real numbers and $a\neq0$. Remember that if $a=0$, then $ax^2=0$ and the graph is linear. Let's look at the simplest form, $y=x^2$ where $a=1$ $b=c=0$. We learned little about this equation earlier when we talked about parent function and vertical line test. The graph of this equation is a parabola (U-shaped) opens upward with the vertex at the lowest point. And the graph of the parabola decreases on the interval from negative infinity to zero and increases on the parabola interval from zero to positive infinity. The equation is written as standard form $y=ax^2+bx+c$ where if "a" is positive, the parabola opens upward and if "a" is negative, the parabola opens downward (vertex at the maximum point). It is also written as vertex form $y=a(x-h)^2+k$ where a, h, and k are real numbers and (h,k) is the vertex coordinate point.

Solving quadratic equations by graphing

Step 1. To solve the quadratic equation by graphing, we need to first find the vertex coordinate point (h, k) of the parabola. In the standard form $f(x)=ax^2+bx+c$, the h and k values are found by $h=\frac{-b}{2a}$ $k=f(h)$, while in the vertex form h and k values are in the vertex form $f(x)=a(x-h)^2+k$.

Example 1: $y=3x^2-2x+7$

a=3, b=-2, c=7

the h and k values are found by plugging the a and b values into the $h=\frac{-b}{2a}$ and using k=f(h).

Example 2. $Y=4(x-5)^2+6$

Here, h=5 and k=6

Vertex coordinate point (h, k)=(5, 6)

After finding the vertex point (h, k), the we need to find two more points , ideally, these points are the x-intercepts with the parabola if the graph intersects with the x-axis. X-intercepts are the roots of the quadratic equation where y=0 or $ax^2+bx+c=0$.

Example 3. $y= x^2-7x+12$

Set equation to zero

$x^2-7x+12=0$

Solve the equation by factoring

$x^2-7x+12=0$, find two numbers whose product is 12 and whose sum is -7

$(x-3)(x-4)=0$, then $x-3=0$ and $x-4=0$

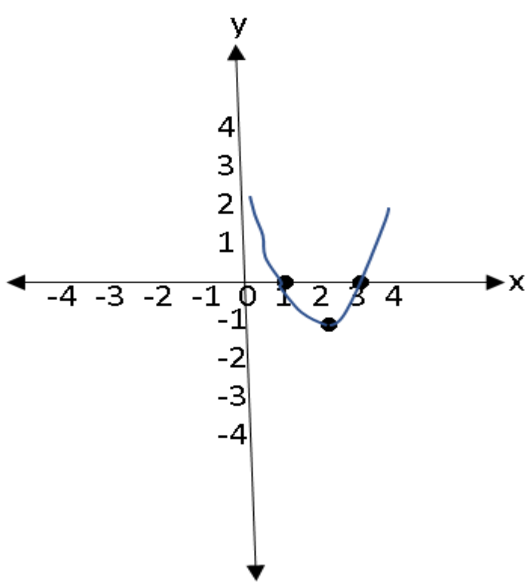
X=3 and x=4 are the points of x-intercepts

Once we find three points on the graph, it is the time to plot the coordinate points and make the graph

Example 4. $y=x^2-2x+1$

$x^2-4x+3=0$, $(x-1)(x-3)=0$, $x=1$ and $x=3$, two coordinate points (1,0) and (3,0) are the x-intercepts. Now, let's find the vertex point.

1. a=1, b=-4, $h=-\frac{-b}{2a}=\frac{-(-4)}{2(1)}=\frac{4}{2}=2$,
2. plug in the h or x value, $k=f(h)=(2)^2-4(2)+3=-1$
3. Plot the graph using vertex point (2,-1) and x-intercepts (1,0) and (3,0)



Solving quadratic equations:

Quadratic equation is a polynomial equation of degree 2. These equations can be solved by factoring, completing the square, using the quadratic formula, and graphing. In the previous section, we learned how to solve a quadratic equation graphically. We look at the other three methods.

Example. Solve the quadratic equation $y=x^2-5x+6$

Method 1. Factoring

Step 1. Set the equation equal to zero.

$x^2-5x+6=0$

step 2. Find two numbers whose product is 6 and whose sum is -5, write the factors with 1th degree x, and then solve the equation.

$(x-2)(x-3)=0$

$x-2=0$, $x=2$, and

$x-3=0$, $x=3$

Method 2: Completing the square

Step 1. Set the equation equal to zero

$x^2-5x+6=0$

Ste- 2. Subtract 6 from both sides

$x^2-5x+6-6=0-6$

Step 3. Add $\frac{25}{4}$ to both sides

$x^2-5x+\frac{25}{4}=\frac{25}{4}-6$

Step 3. Factor the left side and simplify the right side

$(x--\frac{5}{2})^2=\frac{1}{4}$

Step 4. Take square root of both sides

$x-\frac{5}{2}=\pm-\frac{1}{2}$

Step 5. Add $-\frac{5}{2}$ to both sides, simplify, and solve the equation

$x--\frac{5}{2}+\frac{5}{2}=\frac{1}{2}+\frac{5}{2}$, x=3

$x--\frac{5}{2}+\frac{5}{2}=-\frac{1}{2}+\frac{5}{2}$, x=2

Method 3. Using quadratic formula, $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ where a is the coefficient of the x^2 term, b is the coefficient of the x term, and is the constant. In our example, a=1, b=-5, and c=6. Substitute the values for a, b, c and solve the equation.

$\frac{-(-5)\pm\sqrt{(-5)^2-4(1)(6)}}{2(1)}=\frac{5\pm\sqrt{25-244}}{2}=\frac{5\pm1}{2}$, x=3 and x=2

Chapter 15) Exponential equation/function

Exponential growth and decay:

Exponential function is a function with variable in the exponent. It is important in many aspects of daily life including finance and scientific communities. Whether calculating accrued interest in your invested bank account, credit card, car loan, or home loan or studying population growth, it is important to know exponential functions. Originally, exponential growth (increase in the value or quantity) or decay (decrease in the value or quantity) function was written as $y=ab^x$, where :"a" represents the initial value or quantity, "b" represents growth factor when b=1+r (b>1) and "r"is the growth rate, and "b" also represents decay factor when b=1-r and r is the decay rate, and x is the independent variable. The exponential equation is also written in the $y=a(1+r)^x$ and $y=a(1-r)^x$ for exponential growth and exponential decay, respectively.

Example 1: Suppose the population of alligator for a small town in the state of Florida was 200 in 2010. What is the alligator population in 2015 if population growth is 6%.

$y=ab^x$, $a=200$, $b=1+r=1+0.05=1.05$, variable x is time from 2010-2015 which is 5 years

$y=200(1.05)^5=255$ number of alligators in 2015

Example 2. The value of a car was $20,000 in 2003. How much is the value of the car in 2013 if it depreciated 7% annually?

$y=ab^x$, $a=20,000$, $b=1-r=1-0.07=0.93$, x= time from 2003-2013=10

$y=20000(0.93)^{10}=9679$ value of the car in 2013

Example 3. John opened a saving account with $5000 in 2000. What is his total account value in 2015 if his saving account accrues 5% interest annually?

$y=a(1+r)^x$, a=$5000, r=5%=0.05, x=t=2000-2015=15

y=5000(1+0.05)15=5000(1.05)15= $10394.64 amount after 15 years

Another form of exponential function is represented by $A(t)=A_oe^{kt}$ where "A" is the value at time "t","A_o" is the initial value, "k" is the growth rate (k>0 denotes increasing growth and k<0 denotes decreasing growth), exponential "e" is the Euler's number=2.718283000, and 't' is the time. This is used to model situations when unlimited availability of food allows continuous exponential population growth in animal kingdom. For instance, bacterial growth is represented by this exponential function. Also, continuous radioactive decay in radioactive elements such as Phosphorus-32 (P-32) and Sulfur-35 (S-35) is represented by this form.

Example 1. Bacterial population in our intestine double every 20 minutes.

Starting with 100 of these bacteria, what is the number of bacteria after 48 minutes?

Step 1. Find the k value

$A(t)=A_oe^{kt}$, $2=1e^{k20}$, take natural log of both sides, $\ln2=\ln e^{20k}$, make 20k the coefficient of the lne, $\ln2=20k \ln e$, we know $\ln e=1$, then $\ln2=20k$, divide both sides by 20, $\ln2/20=20k/20$, $\ln2/20=k$, $k=0.034657359$

Step 2. Plug in the k value

$A(t)=(100)e^{(0.034657359)(48)}$ = 528 bacteria after 48 minutes

In the following examples, we will be using $A(t)=A_oe^{kt}$ formula to solve problems working with radioactive decay according to exponential model. Half-life of a radioactive material is the time taken for a substance to decay 50%.

Note: Using the exponential form $A(t)=A_oe^{kt}$, we can substitute A for $\frac{1}{2}A_o$ (showing remaining substance after one half-life) and solve for k (decay rate):

$\frac{1}{2}A_o=A_oe^{kt}$, $\frac{1}{2}=e^{kt}$, take natural logarithm from both sides, $\ln\frac{1}{2}=\ln e^{kt}$, $\ln\frac{1}{2}=kt \ln e$, we know $\ln e=1$, $\ln\frac{1}{2}= kt$, and then $k=(\ln\frac{1}{2})/t$ and $t=(\ln\frac{1}{2})/k$ where k is the decay rate and t is the half-life (these are useful expressions in problems dealing with radioactive decay)

Example 2. The half-life of sulfur-35 is about 87 days. How much of a sample of 300 gram sample will be left after 600 days?

Let'600s start with substituting the values in the formula

We know $k=(\ln\frac{1}{2})/t$, therefore $k=(\ln\frac{1}{2})/(87)=-0.008$, and $A(t)=A_oe^{kt}$, $A(600)=300e^{((\ln1/2)/87)(600),}$ $A(600)=300e(-0.008)(600)=300e-4.8=4.33$ grams

Example 3. Half-life of phosphorus-32 is 14.29 days. How much time has passed for a sample of 150 grams that has decayed to 20 grams?

$A(t)=A_o e^{kt}$, $k=(\ln\frac{1}{2})/t$ = $(\ln\frac{1}{2})/(14.29)$ =-0.049

$20=150e^{(-0.049)t}$, divide both sides by 150, simplify, and take natural logarithm,

$\ln 2/15=\ln e^{(-0049)t}$, $\ln 2/15=-0.049t$, t=41.7 days

Example 4. A 6 kg sample of carbon-14 decays by 0.35 kg in 537 years. What is the half-life of carbon?

Find the k by plugging in the values, $5.65=6e^{k(537)}$ divide both sides by 6, $5.35/6=e^{537k}$, take natural logarithm of both sides, $\ln(5.35/6)=\ln e^{537k}$, $\ln(5.35/6)=537k$,

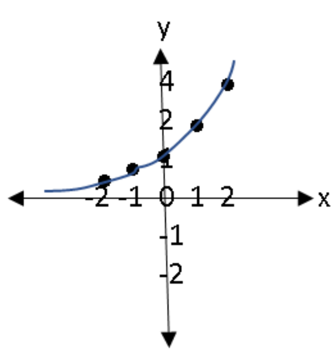
k=-0.00012, use the expression t=ln(1/2)/k=t to find the t value, ln(1/2)/(-000012)=t, t=5776 years

Graphs of exponential functions:

To begin graphing of exponential function, we start with constructing a table of values, then plotting points on a coordinate plane, and finally drawing graph by connecting the points.

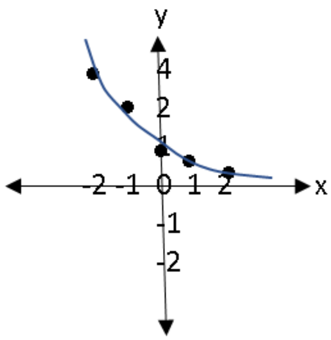
Example 1. $y=2^x$ This is an exponential growth where the x values increase toward positive infinity on the right, decrease toward negative infinity on the left (never becomes zero and x-axis is asymptote).

x	y
-2	¼
-1	½
0	1
1	2
2	4



Example 2. $y=(1/2)^x$, This is an exponential decay where the values of x increase toward positive infinity on the left and decrease toward negative infinity on the right.

x	y
-2	4
-1	2
0	1
1	1/2
2	1/4



Geometric sequences:

Geometric sequence or geometric progression is a sequence of numbers where term is determined by multiplying previous one by a fixed, non-zero number called the common ratio, meaning the ratio of two consecutive numbers is constant. For instance, the sequence 3, 6, 12, 24, 48, … is a geometric sequence with common ratio 2. Similarly, the sequence -81, 27, -9, 3,…is a geometric sequence with common ratio $-\frac{1}{3}$. In general, a geometric sequence is written as a, ar, ar^2, ar^3, ar^4, …where a is the first number and $r\neq0$ is the common ratio where it can be positive or negative. The n^{th} term of geometric sequence is given by explicit formula $a_n=ar^{n-1}$ where a is the initial value and r is the common ratio. It is also written as recursive formula $a_n=ra^{n-1}$. To check whether a given sequence is a geometric sequence, we check whether the consecutive terms in the sequence have a constant ratio.

Example 1. What is the seventh term of the sequence 768, -192, 48, -12, 3, …?

Step 1. Find the common ratio and the common difference in order to determine whether the sequence is geometric or arithmetic.

Common ratio is constant at –1/4 and it is a geometric sequence.

Step 2. Use the previous term and the common ratio to find the seventh term:

Sixth term: 3×(-1/4)= - $\frac{3}{4}$

Seventh term: (- $\frac{3}{4}$)(-$\frac{-1}{4}$)=$\frac{3}{16}$

Or use the explicit formula: a_7=(768)((-1/4)$^{7-1}$)= $\frac{3}{16}$

Example 2. What is the sixth term in a geometric sequence defined $a_n=3a_{n-1}$ and a_1=1/3?

We know the sequence is defined by recursive formula where the common ratio is 3 and the first number is 1/3. Therefore, we can use the explicit formula: $a_6=(1/3)(3)^{6-1}=81$

Example 3. What is the seventh term in a geometric sequence defined by $a_n=4(-1/2)^{n-1}$?

$A_7=4(-1/2)^{7-1}=128$

Chapter 16) Logarithmic and exponential functions

Introduction/exponential and logarithm functions:

Logarithm was introduced by John Napier in the 17th century as means to simplify calculations and later on, was adopted by navigators, scientists, and astronomers to ease computations using multi-digit numbers. Logarithm is the inverse of the exponentiation and vice versa. Exponentiation is raising a positive real number (b) to a power of real number “y” to produce a positive number “x” **($x=b^y$)**. Logarithm is the nth power of a number “b”, called base to another number “y” to obtain a positive number “x” which is expressed as log_b **(x)=y (b>0, b≠1, x>0)**. For example, the base 3 logarithm of 81 is 4 as 3 to the power of 4 is equal to 81 ($81=3^4=3\times3\times3\times3$) and denoted as $\log_3(81)=4$ (it is as logarithm of 81 to base 3 or base 3 logarithm of 81 equals 4) and $81=3^4$. The logarithm to base 10, e (2.718), and 2 are the most commonly used logarithms are called common logarithm (base-10 log denoted as $log_{10}\,(x)$ or $log\quad(x)$), natural logarithm (base-e logarithm denoted as ln (x) or log_e(x)), and binary logarithm, respectively.

Example 1: log_2(128), ask to what power does the base 2 has to be raised to produce 128. The 128=2×2×2×2×2×2×2=2^7. Therefore, log_2(128)=7

Example 2: log_3(243). We know 243=3×3×3×3×3=3^5. Therefore, log_3(243)=5.

Example 3: log_4(256). Again, ask yourself that to what power does the base 4 needs to be raised to produce the number 256. We know 256=4×4×4×4=4^4. log_4(256)=4.

Example 4: log_5(625). 625=5×5×5×5=5^4. Therefore, log_5(625)=4.

Example 5: log_6 (1296). 1296=6×6×6×6=6^4. Therefore, log_6 (1296)=4.

Example 6: log_7 (343). 343=7×7×7=7^3. log_7 (343) =3.

Example 7: log_8 (4096). 4096=8×8×8×8=8^4. Therefore, log_8 (4096) =4.

Example 8: log_9 (729). 729=9×9×9=9^3. log_9 (729) =3.

Example 9: log_{10} (1000). 1000=10×10×10=10^3. Therefore, log_{10} (1000) =3.

Example 10: $log_{10}\,(\frac{1}{1000})$. $\frac{1}{10^3}$ =10^{-3}. Therefore, $log_{10}\,(\frac{1}{1000})$=-3.

Example 11: $log_{\frac{1}{3}}$ (243). 243=3×3×3×3×3=3^5 . here, the base is a fraction which is $\frac{1}{3}$ =3^{-1}.

Therefore, $log_{\frac{1}{3}}$ (243)=-3.

Example 12: $log_{6/5}$ ($\frac{1296}{625}$). Here, we have two fractions. As before, we have to find out that

to what power does the base $\frac{6}{5}$ needs to be raised to produce the fraction $\frac{1296}{625}$..

$\frac{1296}{625}$=$\frac{6^4}{5^4}$ =($\frac{6}{5}$)^4. Therefore, $log_{6/5}$ ($\frac{1296}{625}$)=4.

Practice problems:

Problem 1. log_{625} (5) =$\frac{1}{4}$

Problem 2. log_5 ($\frac{1}{625}$)=-4

Problem 3. log_{343} (7)=$\frac{1}{3}$

Problem 4. log_7 ($\frac{1}{343}$)=-3

Problem 5 log_{343}.($\frac{1}{7}$)=- $\frac{1}{3}$

Problem 6. log_{625} ($\frac{1}{5}$)=- -$\frac{1}{4}$

Logarithmic identities or logarithmic laws:

1. log_b (b)=1
2. log_b (1)=0
3. log_b(b^x) =x (b could be all real numbers)

4. $b^{log_b(x)}$ =x (b>0)

5. $log_b b$ ($\frac{x}{y}$)=log_b(x) - log_b (y)

Example: log_5 ($\frac{625}{25}$)=log_5 (625)- log_5 (25)=4-2=2

6. log_b (xy)=log_b (x) +log_b (y)

Example: log_6 (1296) =log_6 (36) (36)=log_6 (36)+ log_6 (36)=2+2=4

7. log_b (x)^n =n log_b (x)

Example: log_3 (81)=log_3 (3^4)= 4log_3 (3)=4

8. log_b($\sqrt[n]{x}$)= $\frac{log_b(x)}{n}$

Example: log_7 ($\sqrt[7]{343}$) =$\frac{log_7(343)}{7}$ = $\frac{3}{7}$

9. log_b(x)= $\frac{log_a(x)}{log_a(b)}$, log_b(x)= $\frac{log_{10}(x)}{log_{10}(b)}$, log_b(x)= $\frac{log_e(x)}{log_e(b)}$

Solving logarithmic and exponential equations:

let’s review the following different types of logarithmic and exponential functions together. Remember that we always need to check for extraneous solutions.

Example 1: log_x (64)=2, x^2=8^2, x=8

Example 2: log_a (81)=3, 3^3=a^3 ,a=3

Example 3:log_3 (x-1)=2, x-1=9, add 1 to both sides, x=10

Example 4: log_2 (y+5)=0 , y+5=1, y=-4

Example 5:log_2 (2x-3) - log_2 (x-2)=2, log_2 ($\frac{2x-3}{x-2}$) =2, $\frac{2x-3}{x-2}$ =4, cross multiply, (x-2)4=2x-3,

4x-8=2x-3, subtract 2x and add 8 to both sides,

2x=5, x=$\frac{5}{2}$

Example 6: log_3 (x-1)+log_3 (x+1)=1, log_3 ((x-1)(x+1))=1, (x-1)(x+1)=3,x^2 -1 =3, x^2=4, x=±2, check the solutions,

log_3 (2-1)+log_3 (2+1)=1, solution x=2 is defined

log_3 (-2-1)+log_3 (-2+1)=1, solution x=-2 is undefined

Example 7: log_2 (6-2x)=log_2 (x^2-3x), in the same base logarithm indicates the same expressions within the logarithm. Therefore, 6-2x=x^2-3x, add 2x-6 to both sides and simplify,

6-2x+2x-6=x^2-3x+2x-6, x^2-x-6=0, factor the polynomial,

(x-3)(x+2)=0, x-3=0 and x+2=0, then x=3 and x=-2, check the solutions,

log_2 (6-2(-2))=log_2 ($(-2)^2$-3(-2), the solution x=-2 is defined.

log_2 (6-2(3))=log_2 ($(3)^2$-3(3)), the solution x=3 is undefined.

Example 8: ln(2x)=3, 2x=e^3, x=$\frac{e^3}{2}$

Example 9: $4log_4$ (3x)-log_4 (3x)=0,

3log_4 (3x)=0, divide both sides by 3,

log_4 (3x)=0, 3x=1, x=$\frac{1}{3}$

Example 10: ln(y+4)=3, y+3=e^3 , subtract 3 from both sides,

Y=$e^3 - 3$

Example 11: $27^{(x+1)}$ =$81^{(2x+6)}$, rewrite as $3^{3(x+1)}$ =$3^{4(2x+6)}$, the same base must have the same exponents. therefore, 3(x+1)=4(2x+6), distribute 3 and 4.

3x+3=8x+12, subtract 3x+3 from both sides and simplify,

3x+3-3x-3=8x+12-3x-3, 5x+9=0, isolate the variable, 5x=-9, divide both sides by 5, x=-$\frac{9}{5}$

Example 12: Find the domain for the function y=log_{10} (3x-9).

We know that the domain is defined for 3x-9>0. Therefore, 3x>9, x>3, and the domain is defined for any number within (3, ∞).

Example 13: Find the domain for the f(x)=log_e (x^2+5x-14),

The logarithm function is defined for x^2+5x-14>0, factor the inequality,

(x+7)(x-2)>0, then x+7>0, x>-7 and x-2>0, x>2. Since we have two critical points, we have to test them on the number line. Plug in some numbers between the two critical points 2 and -7 such as -2, -4, -7 , and 2 into the original inequality. Check for negative value or zero which is undefined.

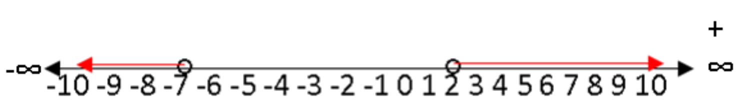
$(-7)^2$+5(-7)-14=0 undefined

$(-4)^2$+5(-4)-14=-18 undefined

$(-2)^2$+5(-2)-14=-20 undefined

$(2)^2$+5(2)-14=0 undefined

Therefore, the domain is defined for x (-∞, -7)U(2, +∞)



Example 14: 3^y =5. Take log_3 from both sides. log_3 (3^y)=log_3 (5),

Y=log_3 (5)=$\frac{log_e(5)}{log_e(3)}$

Practice problems:

Problem 1. Find the inverse function for f(x)=log_e (x-4)+3.

Step 1. Set the function with y and x variables.

y=log_e (x-4)+3

Step 2. Swap the x and the y variable in the function.

x=log_e (y-4)+3

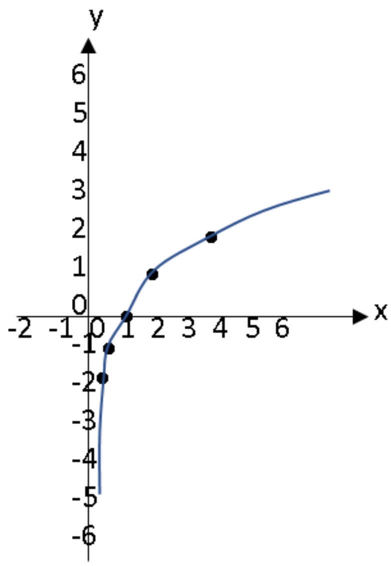
Graph of logarithm and exponential functions:

To graph a logarithm function, the easiest way is to take advantage of the fact that this function is the inverse of the exponential function, that is to say, rewrite the logarithm function in exponential form and then construct a table. Later on, when you feel confident to work with logarithm, you can directly make table using the logarithm function itself. In the following several examples, we will look at two different methods of graphing logarithm function.

Now, let's begin with writing the exponential form (x=b^y) of the logarithm function (y=log_b (x)). And, then plug in some "y" values into the exponential form, find the corresponding "x" values, and complete a table of values. notice that the table formed contains x- and y-coordinates for plotting a logarithm function. Let's begin with the logarithm function y=log_2 (x), write the exponential form x=2^y, plug in the "y" values -2, -1, 0, 1, and 2 into the x=2^y,and find corresponding "x" values. The x- and y-coordinates in this table are the values we use to plot the graph for the logarithm function (see below). Remember that the logarithm is undefined fir x=0 and the negative values of the domain. The graph for the exponential function plotted by swapping the x- and y-coordinates and forming a new table of values (see below).

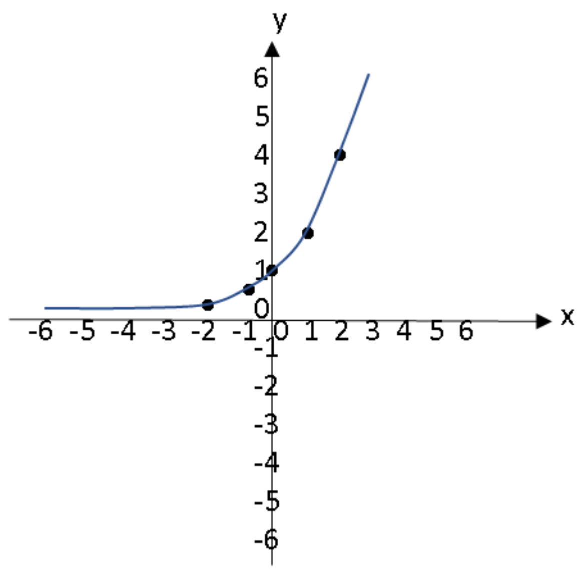
$y = log_2$(x)

x	y
0.25	-2
0.5	-1
1	0
2	1
4	2



x=2^y

x	y
-2	0.25
-1	0.5
0	1
1	2
2	4



Transformation:

Logarithmic function:

The standard formula for logarithm function is y=log_b (x-h)+k.

h: horizontal translation:

h>0: h units to the right

h<0: h units to the left

Has vertical asymptote at x-h=0 and x=h

Has a domain of (h, infinity) and a range of negative infinity to positive infinity.

k: vertical translation:

k>0: k units up on the y-axis

k<0: k units down on the y-axis

a: orientation and slope:

a<0: reflection across the x-axis

IaI>1: expansion vertically

IaI<1: compression vertically

Example 1: y=log_3 (x+1)-1

Solution:

Step 1. Identify a, h, and k values.

a=1, h=-1, k=-1

Step 2. Construct the table of values for parent function y=log_3 (x).

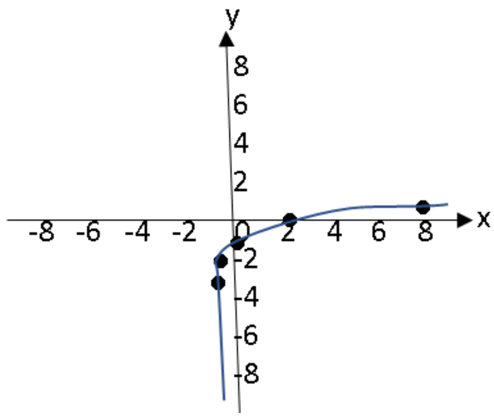
y=log_3 (x)

x	y
1/9	-2
1/3	-1
1	0
3	1
9	2

Step 3. Apply the transformation values for a, h, and k values and make a table of values for function y=log_3 (x+1)-1. Remember that the “h” and the “k” values must be subtracted or added to the x- and y-coordinates in the parent function to find the transformed points. Plot the graph using the new values (see below).

y=log_3 (x+1)-1

x	y
-8/9	-3
-2/3	-2
0	-1
2	0
8	1



Example 2. y=-log_3 (x+1)-1

Solution:

Step 1. Again, identify a, h, and k values.

a=-1, h=-1, k=-1

Notice that the a=-1 but everything else is the same.

Step 2. Construct the table of values for parent function y=log_3 (x).

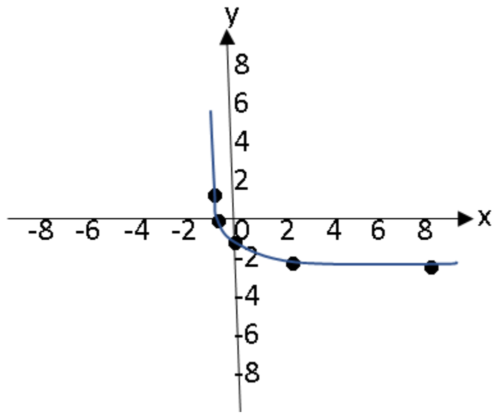
y=log_3 (x)

x	y	reflected y
1/9	-2	2
1/3	-1	1
1	0	0
3	1	-1
9	2	-2

Step 3. Construct a table of values for function y= $-log_3$ (x+1)-1 using reflected y values. . Plot the graph as described in example 1. Remember that we need to use the reflect the y-coordinates before applying k values.

$y=-log_3(x+1)-1$

x	y
-8/9	1
-2/3	0
0	-1
2	-2
8	-3



Example 3: Plot the graph of parent functions $log_{\frac{1}{2}}$ (x), log_2 (x), and log_3 (x).

$log_{\frac{1}{2}}$(x)

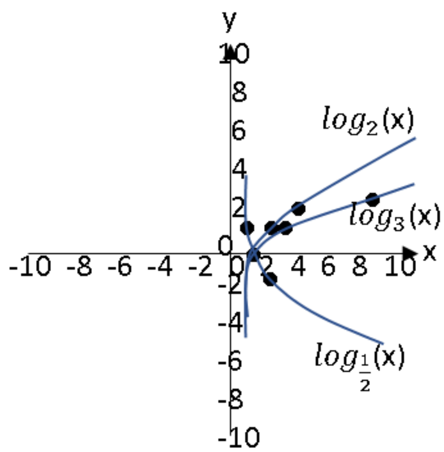
x	y
1	0
2	-1
1/2	1

log_2(x)

x	y
1	0
2	1
4	2

log_3(x)

x	y
1	0
3	1
9	2



Example 4. Graph simple exponential Y=3^x and find the domain and the range.

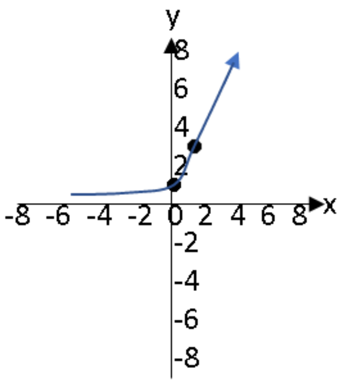
Solution:

Step 1. Construct a table for x=0 and x=1

Step 2. Draw the horizontal asymptote at y=0 (x-axis). Start from horizontal asymptote and plot the graph using the two points.

Step 3. The domain of a exponential function is (-∞,∞) and the range here is (0,∞)

x	y
0	1
1	3



Example 5: Graph the exponential function y=2^{x+2} -2.

Step 1. The graph has shifted 2 units down on the y-axis (horizontal asymptote is y=-2) and 2 units to the left on the x-axis. Graph is in the 1th quadrant since x and y are both positive.   Set x+4 equal to zero and 1 for two points on the graph.

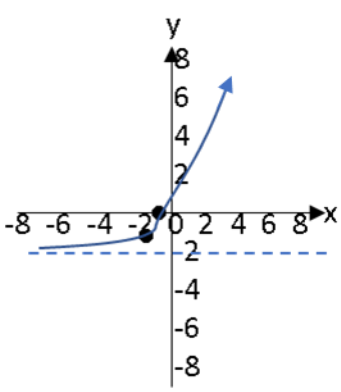
Y=-2 horizontal asymptote

x+2=0, x=--2

x+2=1, x=-1

Step 2. Construct a table and plot the graph

x	y
-2	-1
-1	0



Example 6. Graph and find the domain and the range of the exponential function

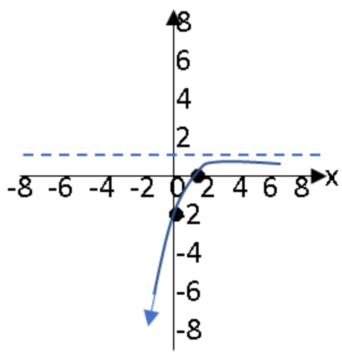
y=$-3^{(1-x)}$ +1

Solution:

Step 1. The graph has shifted 1 unit up on the y-axis (y=1 is the horizontal asymptote) and 1 unit to the left on the x-axis. Y and x ate both negative and the graph will be in the 3^{rd} quadrant.

Step 2. Find two points by setting 1-x=0 and 1-x=1. Construct a table and plot the graph. D=(-∞,∞) and R=(1, ∞)

x	y
1	0
0	-2



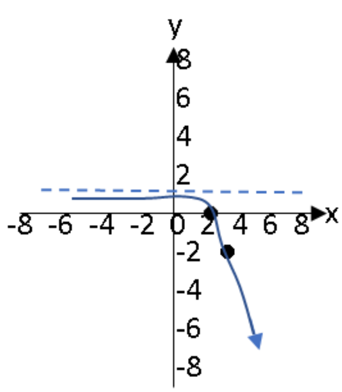
Example 7. Solve the exponential function y=1- 3^{x-2} graphically and fine the domain and the range.

Solution:

Step 1. The graph has shifted 2 units to the right on the x-axis and 1 unit up on the y-axis (y=1 is the horizontal asymptote). The graph is in the 4^{th} quadrant since x is positive and y is negative.

Step 2. Draw the horizontal asymptote and plot the graph using two points. D=(-∞,∞) and R=(1, ∞).

x	y
2	0
3	-2



Step by step review of transformation:

Reflections over the x-axis, y-axis, and the origin:

Let's look at the following functions.

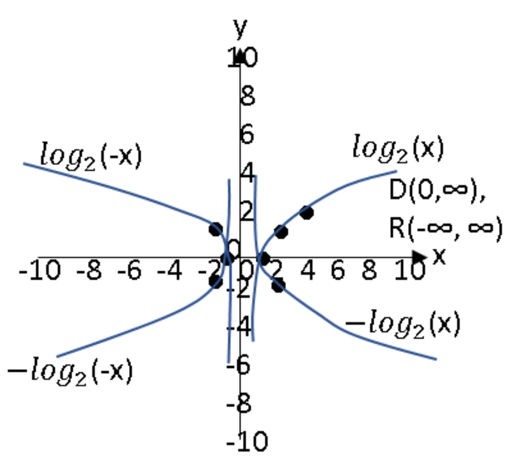
1. -log_2 (x)
2. log_2 (-x)
3. $-log_2$ (-x)

Solution:

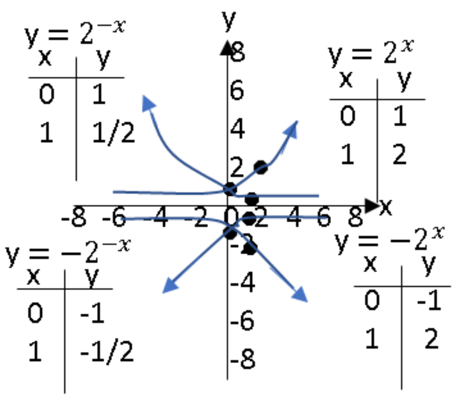
A negative coefficient outside the logarithm function as in -log_2 (x) indicates a reflection across the x-axis. Therefore, the graph will be in the 4th quadrant as shown below.

A negative sign inside the logarithm as in the function log_2 (-x) indicates a reflection over the y-ay-axis and in the 2^{nd} quadrant as shown in the graph.

If there is negative outside and inside the logarithm function, then, the graph is reflected over the origin and in the 3^{rd} quadrant (see below).



Plot the graph of the exponential parent functions y= 2^x , y= 2^{-x} , y= -2^x , and y= -2^{-x}. When x and y are both positive, graph is in the 1th quadrant. Then, a negative y, will reflect the graph over the x-axis. If x and y are both negative, the graph is reflected over the origin. And a negative x will reflect the graph over the y-axis (see below).



Practice problems:

Problem 1. Graph the following functions.

A. Y=-log_2(x-3)+2

B. $y = log_2$ (x+2)+4

C. y=-2log_2 (x+1)+3

D. y=log_2 (2-x)-3

Solution guide:

If you feel confident about working with logarithm rules, try to solve the above functions graphically, a little differently, by finding the vertical asymptote and two points, otherwise, I suggest to rewrite the exponential form as explained earlier. For instance:

$y = log_2$ (x+2)+4, set x+2=0 x=-2 is the vertical asymptote and two more points by setting the x+2=1, x=-1 and then setting x+2=2, x=0. Plug in the x values -1 and into the equation and find y values. finally, construct a table and plot the graph using the vertical asymptote and the two points.

Practice 2. Find the followings.

1. 6^{x-7}=216, check for possible rewriting of the right side into an exponential form

6^{x-7}=6^3,   x-7=3, x=10

2. 3^{a+4}=15, take base-3 logarithm from both sides

$log_3^{3^{a+4}}$ =log_3 (15), a+4=log_3 (3) +log_3 (5),

a+4=1 +log_3 (5), subtract 4 from both sides,

a= log_3 (5) -3

3. log_2 (log_2(y))=2, raise both sides to base-2,

$2^{log_2\ (log_2(\mathrm{y})),}$=$2^2$,

$log_2(\mathrm{y}) = 4$,

Y= 2^4=16

4. 3log_{10}(x)=1, divide both sides by 3

log_{10}(x)=$\frac{1}{3}$,

X=$10^{\frac{1}{3}}$, x=$\sqrt[3]{10}$

Chapter 17) Radical expressions and equations

Solving radical equations:

Radical equations are equations containing radical expressions. Radical equations are solved by isolating radical expressions on one side of the equation, squaring or taking both sides to higher power, and checking the solutions by plugging into the original equation. Checking solution is a necessary step to identify true solution and extraneous solution.

Example 1. $\sqrt{x}$ -1=2

Step 1. Isolate radical expression by adding 1 to both sides

$\sqrt{x}$ -1+1=2+1, $\sqrt{x}$=3

Step 2. Square both sides to get rid of radical expression, remember that we are only dealing with principal square root (non-negative value), X=9

Step 3. Check to find the true solution by plugging the x value into the original equation

$\sqrt{9}$ -1=2, 3-1=2, it is a true solution

Example 2. $\sqrt{3y+4}$ -3=2

Step 1. Add 3 to both sides to isolate radical expression

$\sqrt{3y+4}$ -3+3=2+3, $\sqrt{3y+4}$ =5

Step 2. Square both sides to get rid of radical expression

3y+4=25

Step 3. Subtract 4 from both sides and find the y value

3x+4-4=25-4, 3x=21, x=7

Step 4. Check the y value

$\sqrt{3(7)+4}$ -3=2, 5-3=2, it is a true statement and y=7 is a true solution

Example 3. $4\sqrt{2a-3}$ -1=$3\sqrt{2a-3}$ +6

Step 1. Subtract $3\sqrt{2a-3}$ from both sides, and then add 1 to both sides to isolate radical expression:

$4\sqrt{2a-3}$ -1-$3\sqrt{2a-3}$ +1=$3\sqrt{2a-3}$ +6- $3\sqrt{2a-3}$ +1

$\sqrt{2a-3}$ =7

Step 2. Square both sides to get rid of radical

2a-3=49

Step 3. Add 3 3 to both sides and divide both sides by 2 to solve the equation

2a-3+3=49+3, 2a=52, a=26

Step 4. Check the value for a in the original equation

$4\sqrt{2(26)-3}$ -1=$3\sqrt{2(26)-3}$ +6

4(7)-1=3(7)+6, 27=27, the statement is true and the solution is true solution

Example 4. $\sqrt{3x+4}$ =x+2

Step 1. Square both sides,

$(\sqrt{3x+4})^2=(x+2)^2$, $3x+4=x^2+4x+4$

Step 2. Subtract (3x+4) from both sides,

$3x+4-3x-4=x^2+4x+4-3x-4$

$X^2+x=0$

Step 3. Factor and solve the equation

x(x+1)=0, x=0 and x=-1

Step 4. Check the solution

$\sqrt{3x+4}$ =x+2, if x=0 then it must satisfy the equation

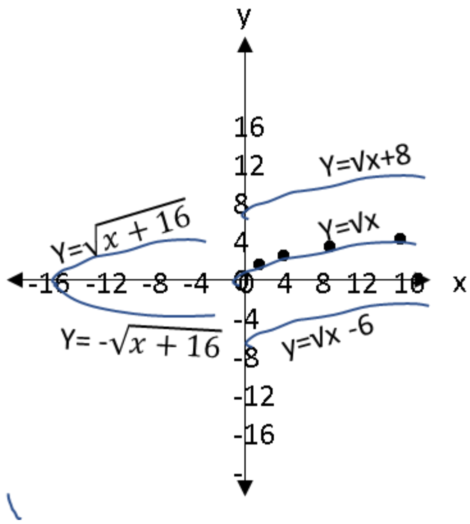
$\sqrt{3(0)+4}$ =(0)+2, it is a true statement and x==0 is a true solution

$\sqrt{3(-1)+4}$ =(-1)+2, it is a true statement and x=-1 is also a true solution

Graphing square root functions:

The parent square root function y=√x where x≥0 and y≥0 is the origin of all other radical functions of the form y= $\sqrt{x+a}$ +b. Transformation of the graph for parent function on the vertical and horizontal axis will generate all other functions of the form y= $\sqrt{x+a}$ +b where negative and positive values of "a" and "b" will move the graph on the horizontal and vertical axis, respectively. Let's start with parent function and move on to other functions. As shown in the following graph, moving y=√x unction 8 units up 6 units down on the vertical axis will form y=√x+8 and y=√x-6 functions, respectively. Notice that the units numbers are written outside radical in the equation. Moving parent function graph on the horizontal axis 16 units to the left has formed y= $\sqrt{x+16}$ and y= $-\sqrt{x+16}$ function. The negative sign in front of the radical expression indicate a flip around horizontal axis. Remember that moving the graph to the left on the x-axis requires a positive number within the radical sign (x+16 within the radical) since x+16 will equal zero (x+16=0 and x=-16) at y=0. Also, moving the graph to the right on the axis requires a number with negative sign. Flipping the graph around vertical axis to create a mirror image requires negative sign before all the terms within radical. For instance, the mirror image of our example shown in the figure below, y= $\sqrt{x+16}$, will be y=$\sqrt{-(x+16)}$ (graph not shown). Stretching or shrinking of these functions is done by putting a number in front of the radical sign. For example, the graph for function y=$4\sqrt{-(x+16)}$ is a stretch for the graph y= $\sqrt{-(x+16)}$ while the graph for the function y=$0.5\sqrt{-(x+16)}$ is a shrink for the graph y= $\sqrt{-(x+16)}$ (data not shown).

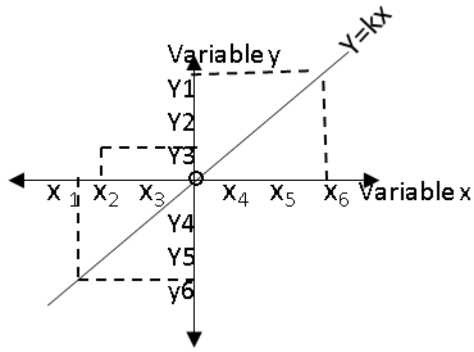
y	x
0	0
1	1
4	2
3	9
4	16



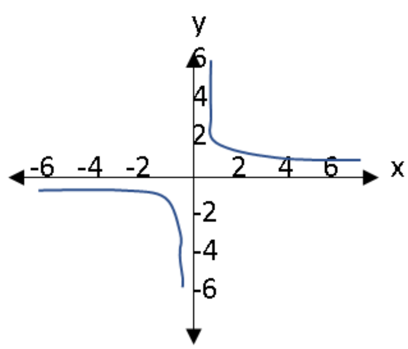
Chapter 18) Rational expressions and equations

Direct and Inverse variation:

Before inverse variation, let's discuss proportionality and variation followed by direct and inverse proportionality and variation. Two variables "y" and "x" are said to be proportional if a change in one is always associated by a change in the other and the changes or variation are related by constant called constant of proportionality or constant of variation "k". The "y" and "x" variables are directly proportional if, for instance, "y" as a function of "x" is always equal to the product of "x" and "k", y=kx (y as a function x where $\frac{y}{x}$ =k is always constant), or equivalently, x=$\frac{1}{k}$ y (x as a function of y where $\frac{x}{y}$ =$\frac{1}{k}$ is always constant with $\frac{1}{k}$ constant of variation). In direct proportionality, when x value increases or decreases, so does the y value accordingly. The equation for direct variation, for instance y as a function of x, is a straight line that passes through the origin with the slope of the constant of variation "k"(see below). For example, circumference of a circle is directly proportional to its diameter with variation constant of π.



If two variables “y” and “x” are inversely proportional (also called varying, inverse variation, or reciprocal proportion), then the product of variables “y” and “x” is non-zero constant “k”, xy=k or y=$\frac{k}{x}$ (y is a function of x), that is to say, when the value for x increases, the value for y decreases and vice versa. For example, the time spent traveling between city A and city B is inversely proportional to the speed of traveling. The graph for two variables with inverse variation is hyperbola on the Cartesian coordinate plane and the products of all “x” and “y” values on the graph are equal to the “k” non-zero constant of variation (graph never crosses either axis).



Graphing rational functions:

In mathematics, rational function is a function that is defined by a rational fraction where the numerator and the denominator are polynomials. The domain of the function is a set of values for which denominator is not zero. To graph a rational function, we have to consider the following steps:

Step 1. Factor the numerator and the denominator to check for the possibility of factors that cancel out. The canceled factors can not be used to find the vertical asymptote when setting the denominator equal to zero. However, we have to solve the canceled factor x value and plug it into the remaining equation (after the similar factors are being canceled) to find the y-coordinate for a point called "graph hole or removable discontinuity" shown as a open dot.

Step 2. Find the vertical asymptote by setting the denominator equal to zero and solving for x values

Step 3. Find the horizontal asymptote using the following rules:

1. Determine the highest degree for the numerator and the denominator.

2. If the highest power of the numerator is smaller than the highest power of the denominator, then the horizontal asymptote is the line y=0 (x-axis).

3. If the highest power of the numerator ties with the highest power of the denominator, the horizontal asymptote is the ratio of the leading coefficient in the numerator over the leading coefficient in the denominator.

4. If the highest power of the numerator is larger than the highest power of the denominator, then there will be no horizontal asymptote but there will be a slant or diagonal asymptote, that is a line y=x and passes through the origin with 45 degrees angle.

Step 4. Find the x and the y intercepts

Step 5. Find a point on either side of the vertical asymptote and use them to do "sign analysis" and determine the actual shape of the graph.

Example 1. Graph the rational function $f(x)=(x-2)/(x^2-2x-3)$

Solution:

Step 1. Factor the denominator

(x-2))/(x-3)(x+1)

tep 2. Find the vertical asymptote

(x-3)(x+1)=0

x-3=0, x=3 and x+1=0, x=-1

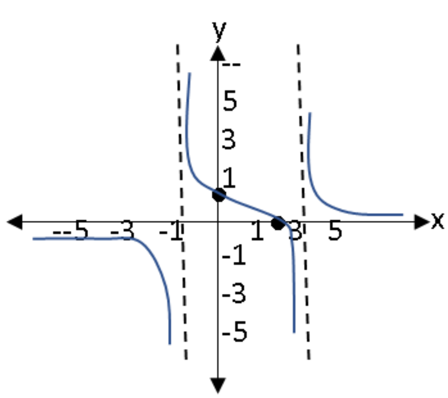
Step 3. Find the horizontal asymptote. In this example, the highest degree for the numerator is smaller than the highest degree for the denominator. Therefore, the line y=0 (x-axis) is the horizontal asymptote.

Step 4. Find the y and the x intercepts

If x=0 then y=2/3 graph must pass through this point

If y=0, then x=2 graph must also pass through this point

Step 5. Do sign analysis using a point on the x-axis before and after each vertical asymptote to find out whether the graph moves toward positive or negative infinity. I used the points x-=4 and x=2.5 around the vertical asymptote at x=3 and the points x=-0.5 and x=-2 around the second vertical asymptote at x=-1 for sign analysis. Below is the graph for this rational function.



Example 2. Graph the rational function $\frac{(x-5)(x+4)}{(x-5)(x+3)}$

Step 1. Here the function is already factored but the (x-5) factor is common in the numerator and the denominator. Therefore, x=5 cannot be used for vertical asymptote. there will be a hole or removable discontinuity in the graph.

Step 2. Find the vertical asymptote

Denominator (x-5)(x+3) must be set equal to zero. However, x=5 can only be used to find the y coordinate for the graph hole.

x-5=0, x=5, substitute in the equation to the y coordinate

y= $\frac{(5+4)}{5+3}$ =$\frac{9}{8}$, notice that we have substituted the x value after canceling out the common factor.

Graph hole point (5, 9/8)

X+3=0, x=-3, used for vertical asymptote

Step 3. Find the horizontal asymptote.

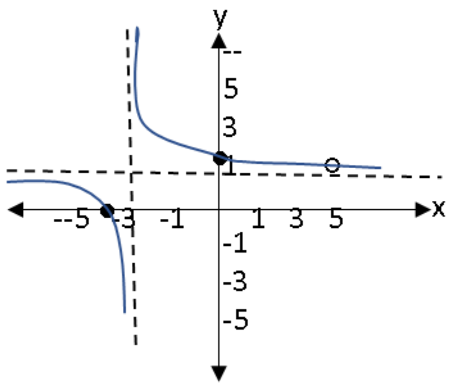
Since the highest power in the numerator ties with the highest power in the denominator, the horizontal asymptote will be the line y=1

Step 4. Find the y and the x intercepts

X=0, then y=4/3

Y=0, then x+4=0, x=-4

Step 5. Do the sign analysis and plot the graph



Example 3. Graph the rational function $\frac{(x-2)(x-4)}{(x-6)}$

Step 1. In this example, the highest power in the number is larger than the highest power in the denominator and there is no common factor. FOIL the numerator and do the polynomial division.

$$\begin{array}{r|l} & x \\ \hline x-6 & x^2-6x+8 \\ & -(x^2-6x) \\ \hline & 8 \end{array}$$

The result is a straight line y=x that passes through the origin with 45° angle and it is slant or diagonal asymptote.

Step 2. Find the vertical asymptote.

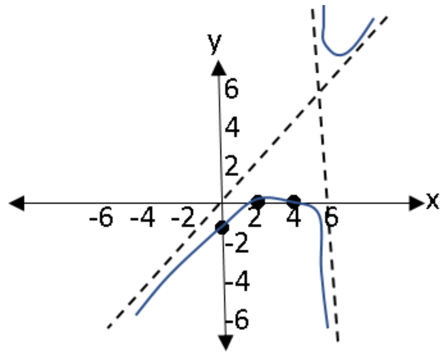
x-6=0, x=6

Step 3. Find the x and the y intercepts.

Y=0, then (x-2)(x-4)=0, x=2 and x=4

X=0, then y=-4/3

Step 4. Do the sign analysis and graph the function



Simplifying rational expressions:

Fractions of rational numbers or numerical fractions such as $\frac{10}{18}$ are simplified by cancelling out common factors in the numerator and the denominator:

$\frac{10}{18}=\frac{2\times5}{2\times9}=\frac{5}{9}$, here, 2 in the numerator is canceled with 2 in the denominator and the fraction is changed to the lowest form. Similarly, rational expressions or polynomial fractions are simplified by finding and canceling shared numerical and polynomial factors in the numerator and the denominator of the rational expression.

Example 1. $\frac{3y-6}{9y-18}$

Look for factors in the numerator and the denominator.

$\frac{3(y-2)}{9(y-2)}=\frac{3(y-2)}{3\times3\times(y-2)}=\frac{1}{3}$, the 3×(y-2) factors in the numerator is shared with the same factors in the denominator and can be cancelled out and simplify the polynomial expression to $\frac{1}{3}$.

Example 2. $\frac{4ab-2a}{2a}=\frac{2a(2b-1)}{2a}$ =2b-1, the cancelled common factor for the numerator and the denominator is 2a.

Example 3. $(9x^2-4)/(9x^2+9x+2)$

Solution:

Step 1. Factor the numerator: (3x+2)(3x-2)

Factor the denominator:

1. Write the 9x as 6x+3x : $9x^2+6x+3x+2$
2. Group the terms: $9x^2+6x+3x+2=(9x^2+6x)+(3x+2)$,
3. Factor 3x out of expression $(9x^2+6x)$: 3x(3x+2)+(3x+2)
4. Factor out (3x+2) : (3x+2) (3x+1)

Step 2. Final factored polynomial fraction: $\frac{(3x+2)(3x-2)}{(3x+2)(3x+1)}$

Step 3. Cancel out (3x+2) and simplify to form $\frac{3x-2}{3x+1}$

Multiplying and dividing rational expressions:

Rational expressions are multiplied by multiplying the numerators with each other and the denominators with other as multiplying fractions of rational numbers. Then, the fraction must be factored and reduced to simplest form. Similarly, to divide rational expressions, flip or reciprocate the expression after the division sign and multiply the rational fractions as in fractions of rational numbers.

Example 1. $\frac{4dy}{3ab} \times \frac{6b}{2d}$

Solution: $\frac{(4dy)(6b)}{(3ab)(2d)} = \frac{(2\times 2dy)(3\times 2b)}{(3ab)(2d)}$, cancel out the same factors and reduce to $\frac{4y}{a}$

Example 2.

$$\frac{2x^2+3x-2}{x^2-4x+4} \div \frac{8x^2-4x}{x^2-4}$$

Step 1. Reciprocate the fraction after division and change the division sign to multiplication sign

$$\frac{2x^2+3x-2}{x^2-4x+4} \times \frac{x^2-4}{8x^2-4x}$$

Step 2. Group terms if necessary and Factor

$$\frac{(2x^2+4x)-(x+2)}{x^2-4x+4} \times \frac{x^2-4}{8x^2-4x}$$

$$\frac{2x(x+2)-(x+2)}{(x-2)(x-2)} \times \frac{(x+2)(x-2)}{4x(x-2)}$$

$$\frac{(2x^2+4x)-(x+2)}{x^2-4x+4} \times \frac{x^2-4}{8x^2-4x}$$

$$\frac{(x+2)(2x-1)}{(x-2)(x-2)} \times \frac{(x+2)(x-2)}{4x(x-2)}$$

Step 3. Cancel similar factors and reduce

$$\frac{(x+2)(x+2)(2x-1)}{4x(x-2)(x-2)}$$

Adding and subtracting rational expressions:

Adding and subtracting of rational expressions with like denominators is done by writing the denominator and adding or subtracting all numerators of the rational expressions. When the denominators are different, we find the least common denominator (LCD) for all rational expressions and proceed as described above.

Example 1. $\frac{3x-2}{2-x} + \frac{x-1}{2-x} = \frac{3x-2+x-1}{2-x} = \frac{4x-3}{2-x}$

Example 2. $\frac{a+1}{3a-2} + \frac{a-1}{a}$

Step 1. Find LCD, divide LCD by each numerator, multiply the result by the corresponding numerator and add them all together, $\frac{a(a+1)+(3a-2)(a-1)}{a(3a-2)}$

Step 2. Add the numerator's terms and simplify

$$\frac{a^2+a+3a^2-2a-3a+2}{a(3a-2)} = \frac{4a^2-4a+2}{a(3a-2)}$$

Example 3. $\frac{2k}{3x+2}-\frac{y}{5}=\frac{(2k)(5)-y(3x+2)}{(3x+2)5}=\frac{10k-3xy-2y}{(3x+2)5}$

Mixed expressions and complex fractions:

When algebraic fractions are mixed with monomial terms, the result is called mixed expression. For instance, the expression $3+\frac{1}{y+2}$ is a mixed expression. A related subject is complex fraction and that is when you are given an algebraic fraction containing one or more fraction in the numerator and/or the denominator. The following expression is an example of a complex algebraic fraction:

$$\frac{\frac{1}{x^2-1}}{\frac{1}{x-1}}$$

Example 1. $3+\frac{1}{y+2}$

Solution: To find a common denominator, multiply the numerator and the denominator of the term 3 (remember that the denominator is 1) by (y+2).

$\frac{3(y+2)}{y+2}+\frac{1}{y+2}$, distribute and add the numerators, $\frac{3y+7}{y+2}$

Example 2. $3-\frac{5}{a}$

Solution: multiply the numerator and the denominator of 3 by a.

$\frac{3a}{a}-\frac{5}{a}$, subtract the numerators, $\frac{3a-5}{a}$

Example 3. $\frac{2x}{3x+y}+x$

Solution: multiply the numerator and the denominator of the term x by (3x+y)

$\frac{2x}{3x+y}+\frac{x(3x+y)}{3x+y}$, since the denominators are, we can add the numerators

$\frac{2x+x(3x+y)}{3x+y}$, distribute,

$$\frac{3x^2+2x+xy}{3x+y}$$

Example 4. (b-2) $+\frac{b+4}{b-2}$

Solution: multiply the numerator and the denominator of the binomial (b-2) to find a common denominator

$\frac{(b-2)(b-2)}{b-2} + \frac{b+4}{b-2}$, add the numerators and simplify, if ncessary.

$\frac{(b-2)(b-2)+b+4}{b-2}$, factor in reverse and add

$$\frac{b^2-3b+8}{b-2}$$

Example 5.

$$\frac{d^2-4}{\frac{4}{d+2}+\frac{1}{d}}$$

Solution: find the common denominator for the denominator of the given complex fraction by multiplying the numerator and the denominator of the rational fractions $\frac{4}{d+2}$ and $\frac{1}{d}$ by d and (d+2), respectively.

$$\frac{d^2-4}{\frac{4d}{(d+2)d}+\frac{d+2}{d(d+2)}}$$

Continue working on the denominator until it is converted into a single rational fraction. Now that there is a common denominator, we can add the numerators.

$$\frac{\frac{d^2-4}{1}}{\frac{5d+2}{d(d+2)}}$$

Write the complex fraction as

$$\frac{d^2-4}{1} \div \frac{5d+2}{d(d+2)}$$

Continue as previously described for division of rational fractions. Reciprocate the fraction after division sign and change the division sign to multiplication sign. It is the final form since there is no common factor to be cancelled out.

$$\frac{d^2-4}{1} \times \frac{d(d+2)}{5d+2} = \frac{d(d+2)(d^2-4)}{5d+2}$$

Example 6.

$$\frac{\frac{xy^2}{d^2}}{\frac{x^2y}{d}}$$

Write the complex fraction and then, continue as you learned how to divide rational fractions and simplify

$$\frac{xy^2}{d^2} \div \frac{x^2y}{d} =$$

$$\frac{xy^2}{d^2} \times \frac{d}{x^2y} = \frac{y}{dx}$$

Solving rational equations:

There are two ways to solve rational equations. One way is to find a common denominator and then, since we know the denominators are equal, the numerators must be equivalent and solved for variable. A second way is to multiply both sides of the equation by the common denominator, and change the rational equation to polynomial equation for solution. Always remember to check your solution for extraneous answer.

Example 1. $\frac{x}{5}$- $\frac{1}{5x}$=$\frac{3}{5x}$

Solution: Multiply both sides by common denominator (5x) and change the rational fraction to polynomial fraction.

$$\frac{x(5x)}{5}-\frac{5x}{5x}=\frac{3(5x)}{5x}$$

Cancel similar factors

x^2-1=3,

Add 1 to both sides

x^2=4, x=±2

Check the solution

$\frac{2}{5}$- $\frac{1}{5(2)}$=$\frac{3}{5(2)}$, x=2 satisfies the equation,

$\frac{-2}{5}$- $\frac{1}{5(-2)}$=$\frac{3}{5(-2)}$, x=-2 **does not** satisfy the equation

Example 2. $\frac{3x-1}{x-1}$ =$\frac{2x+3}{x-1}$

Solution: Here since the denominators are equal on both sides if the equal sign, the numerators must be equivalent.

3x-1=2x+3,

subtract 2x from both sides, then add 1 to both sides.

3x-1-2x+1=2x+3-2x+1, x=4,

Check the solution

$\frac{3(4)-1}{4-1}=\frac{2(4)+3}{4-1}$, $\frac{11}{3}=\frac{11}{3}$, solution satisfies the equation

Example 3. $\frac{a}{4}+\frac{3}{4}=-3$

Solution: 4 is a common denominator for fractions in the left side of the equal sign. Let's add them up.

$\frac{a+3}{4}=-3$,

multiply both sides by 4 and then subtract both from 3

A+3=-12, a=-15

Check the solution in the original equation

$\frac{-15}{4}+\frac{3}{4}=-3$, $\frac{-12}{4}=-3$, solution satisfies the equation

Practice questions:

1. $\frac{a}{4}+\frac{3}{4}=\frac{-a}{6}$
2. $\frac{-4}{y}+\frac{3}{4y}=-3$
3. $\frac{5}{b-2}=5-\frac{4}{b-4}$
4. $\frac{2}{5x}-\frac{x}{3}=\frac{1}{x}$
5. $\frac{3}{4x}-\frac{2}{3x}=\frac{4x}{7}$
6. $\frac{x^2-6x+15}{3x-4}=\frac{x+3}{3x-4}$

Chapter 19) Trigonometry and geometry

Trigonometry:

Angle and trigonometric identities

Trigonometric identities are equations that involve trigonometric functions. These equations are true for all defined values of the variables on both sides of the equality. The trigonometric identities deal with certain functions of one or more angles and they are different from geometric triangle identities where angle and side lengths are involved.

Most common identities are:

1. Sin θ=1/csc θ

2. cos θ=1/sec θ

3. tan θ=sin θ/cos θ=1/cot θ

4. cot θ=cos θ/sin θ=3=1/tan θ

5. sec θ=1/cos θ

6. csc θ=1/sin θ

7. $\sin^2\theta + \cos^2\theta = 1$

8. tan(-θ)=-tan θ

9. sin(-θ)=-sin θ

10. cos(-θ)=cos θ

Angle sum and difference identities:

1. sin (α+ β) =sin α cos β +cos α sin β

2. sin (α- β) =sin α cos β -cos α sin β

3. cos (α+ β) =cos α cos β -sin α sin β

4. $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$
5. $\tan(\alpha+\beta)=(\tan\alpha+\tan\beta)/(1-\tan\alpha\tan\beta)$
6. $\tan(\alpha-\beta)=(\tan\alpha-\tan\beta)/(1+\tan\alpha\tan\beta)$
7. $\cot(\alpha-\beta)=(\cot\alpha\cot\beta+1)/(\cot\beta-\cot\alpha)$
8. $\cot(\alpha+\beta)=(\cot\alpha\cot\beta-1)/(\cot\beta+\cot\alpha)$
9. $\csc(\alpha+\beta)=(\sec\alpha\sec\beta\csc\alpha\csc\beta)/(\sec\alpha\csc\beta+\csc\alpha\sec\beta)$
10. $\csc(\alpha-\beta)=(\sec\alpha\sec\beta\csc\alpha\csc\beta)/(\sec\alpha\csc\beta-\csc\alpha\sec\beta)$
11. $\sec(\alpha+\beta)=(\sec\alpha\sec\beta\csc\alpha\csc\beta)/(\csc\alpha\csc\beta-\sec\alpha\sec\beta)$
12. $\sec(\alpha-\beta)=(\sec\alpha\sec\beta\csc\alpha\csc\beta)/(\csc\alpha\csc\beta+\sec\alpha\sec\beta)$

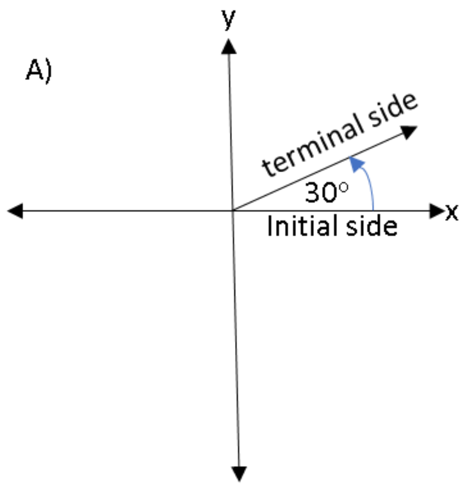
Double angle identities

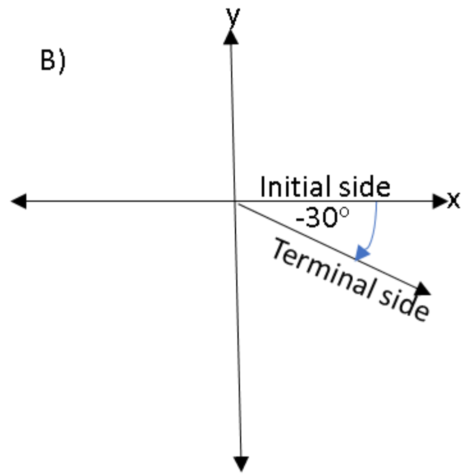
$\mathrm{Sin}2\alpha=2\sin\alpha\cos\alpha$

$\mathrm{Cos}2\alpha=\cos^2\alpha-\sin^2\alpha=1-2\sin^2\alpha=2\cos^2\alpha-1$

In geometry, you learned that an angle is formed by two rays meeting at a point called **vertex**. In trigonometry, this angle is placed on the Cartesian coordinate axis . One ray of the angle is always placed on the positive arm of the x-axis and is called the **initial side** while the vertex is always placed right at the origin. The other ray is called the terminal side. This type of positioning of the angle on the coordinate plane is called **standard position**. When angles are drawn in standard positions, they have direction: it is a positive angle when drawn counterclockwise and it is a negative angle when drawn clockwise. For instance, below, you see the 30^o angle (A) and the -30^o angle (B). Angles are normally represented by Greek letters alpha (α), beta (β), gamma (γ), or theta (θ) and are measured by different units including degrees, radians, and gradians (gons):

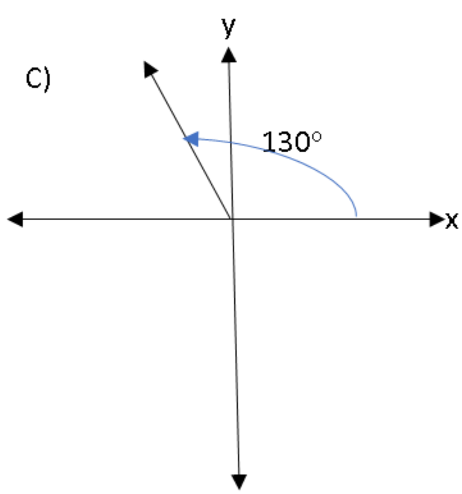
1 full circle turn=360 degrees=2π radians=400 gradians

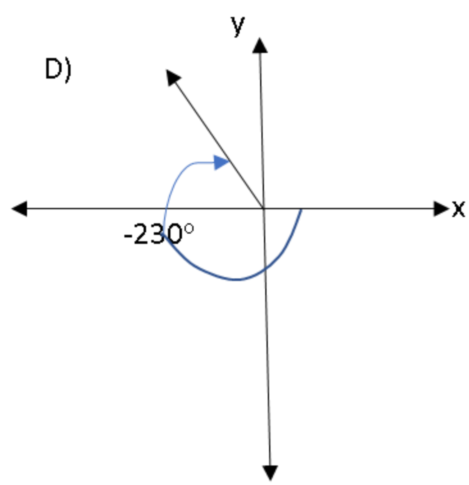




Coterminal angle:

Coterminal angles are angles who have initial sides on the positive arm of the x-axis and have a common terminal side. To find the positive and negative coterminal angles add or subtract 360o (if in degrees) or 2π (if in radians), respectively. Draw two angles, C=130° and D=-230°. Notice that you should draw the D angle clockwise and the C angle counterclockwise. If look at your drawings carefully, you find out that they have the same terminal side, although they represent different angles. Such pairs of angles are called **coterminal angles**.





Example 1: What are the negative and the positive coterminal angles of the angle measured 30^o?

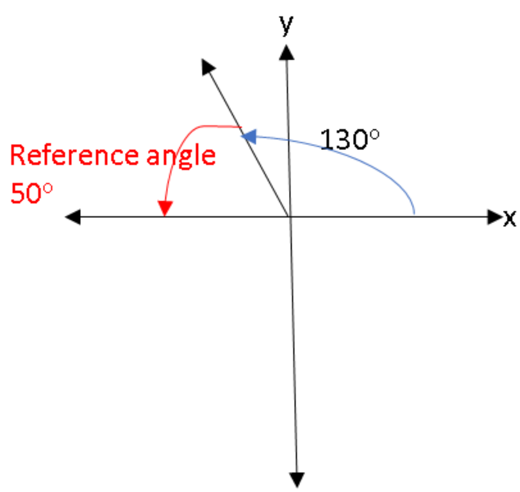
Solution: 30^o-360^o= -330^o, 30^o+360^o=390^o

Example 2: Find the positive and the negative coterminal angles of angle $\alpha=\frac{\pi}{4}$.

Solution: $\frac{\pi}{4}$. -2π=-$\frac{7\pi}{4}$, $\frac{\pi}{4}$.+ 2π = $\frac{9\pi}{4}$

Reference angle:

Reference angle is always a positive angle between 0^o-90^o for every angle drawn in standard position on the coordinate axes. This angle is formed by the terminal side and the x-axis. An angle in standard position and its reference angle is shown below.



Example 1: What is the reference angle for 230^o?

Solution: 230-180=50^o, the reference angle is 50^o

Example 2: What is the reference angle for -150^o?

Solution: Reference angle is always positive:

180-150=30^o, the reference angle is 30^o.

Trigonometric functions for special angles:

Below you see two right triangles. The one on the right is drawn on the is constructed from a 30^o angle in standard position on the unit circle. The two triangles have the same angles and are triangles. Therefore, the corresponding sides are proportional. The value of the hypotenuse for the right triangle is 1 (radius of the unit circle) which is one-half the hypotenuse for the triangle on the left. Correspondingly, other sides of the one on the right are one-half of the one on the left. The side adjacent to the 30^o on the left is √3, consequently, the side adjacent to the 30^o on the right must be √3/2. Similarly, the opposite side to the 30^o on the left is 1 and the opposite side to the 30^o on the right must be 1/2. In the right triangle, the x- and y-coordinates of the point on the unit circle where the terminal side intersects are as follows:

($x=\cos 30^o$, $y=\sin 30^o$).

This is true for any point (x,y) on the unit circle. Now, let's write the six commonly used trigonometric functions.

$$\text{Sin } \theta=\frac{opposite\ side}{hypotenuse}=\frac{y}{1}=y$$

$$\text{Cos } \theta=\frac{adjacent\ side}{hypotenuse}=\frac{x}{1}=x$$

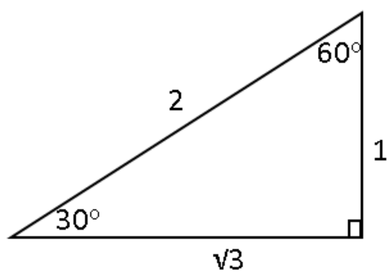
$$\text{Tan } \theta=\frac{opposite\ side}{adjacent\ side}=\frac{\text{Sin }\theta}{\text{Cos }\theta}=\frac{y}{x}$$

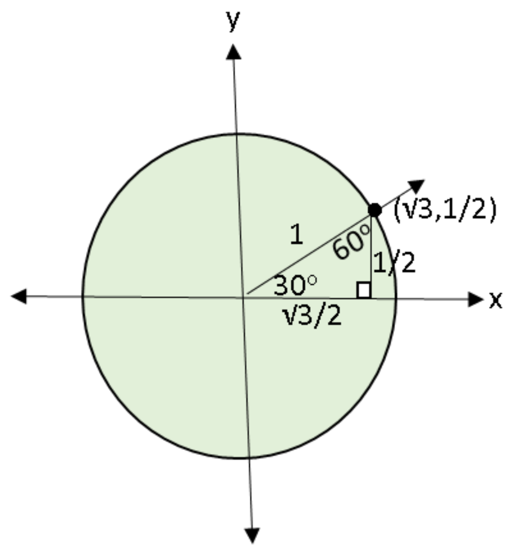
$$\cot \theta=\frac{adjacent\ side}{opposite\ side}=\frac{\text{Cos }\theta}{\text{Sin }\theta}=\frac{x}{y}$$

$$\csc \theta=\frac{hypotenuse}{opposite\ side}=\frac{1}{y}$$

$$\sec \theta=\frac{hypotenuse}{adjacent\ side}=\frac{1}{x}$$

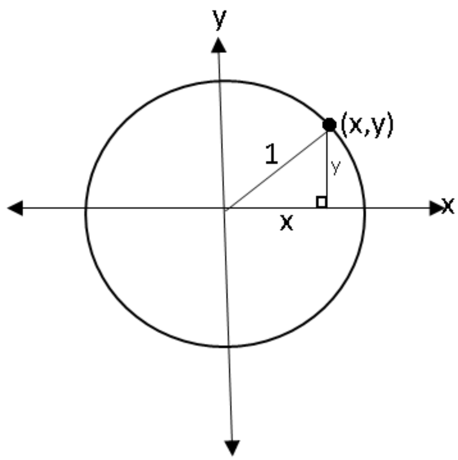
The first two equations indicate that for every point (x,y) on the unit circle (first, second, third, or fourth quadrant) where the terminal side of angle θ in standard position intersects, x-coordinate=cos θ and y-coordinate=sin θ. The six trigonometric equations given above define six trigonometric functions of angle θ anywhere on the unit circle.



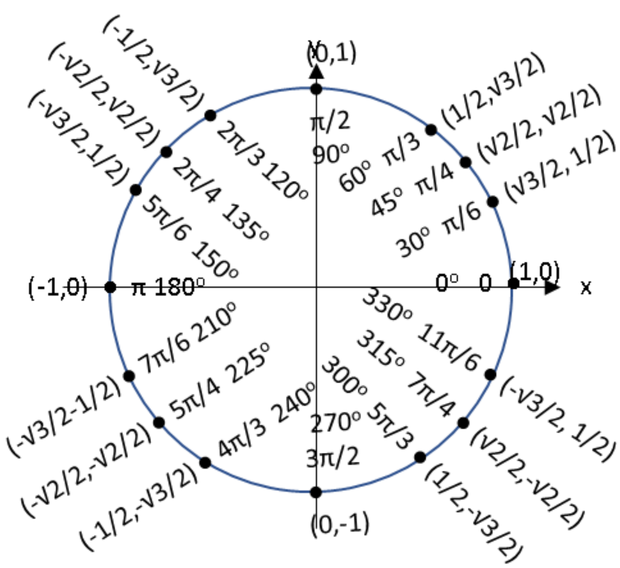


Special angles on unit circle:

A unit circle is a circle that is centered at the origin of a Cartesian coordinate plane with radius 1. For every coordinate point(x_1,y_1) on the circle and the Pythagorean theorem, the equation of unit circle can be deduced as $x^2+y^2=1$ as shown in the diagram below.

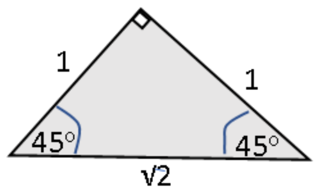


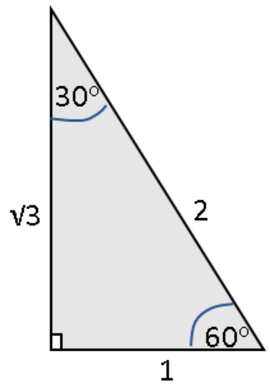
Below sine and cosine functions of special angles are shown around a unit circle.



Special triangles used in trigonometry:

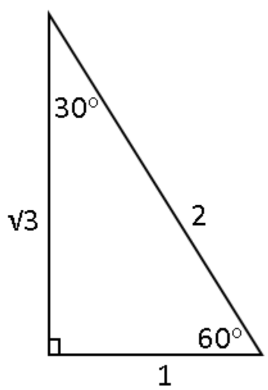
There are two special triangles in trigonometry; the 30°—60°-90° triangle and 45°-45°-90° isosceles triangle. They are called because by a simple geometry we are able to find the length of their sides or the measure of their angles. In a 30°—60°-90° triangle the sides are in ratio 1: 2: √3 that represent the side opposite the 30°, hypotenuse, and the side opposite the 60°, respectively. Also, in 45°-45°-90° triangle, sides are in ratio 1: √2: 1 representing side one, hypotenuse, and side two, respectively. Knowing these ratios allows us to evaluate the functions of 30°, 45°, and 60°. Remember that the larger side is opposite larger angle.





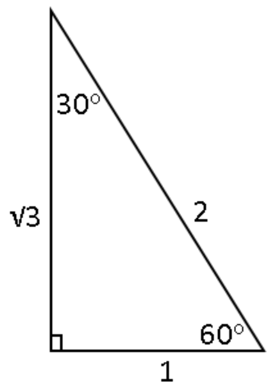
Example 1. Evaluate sin 30°.

Solution:
Step 1. Sketch a triangle and write the ratio numbers on the figure.



Step 2. We know sin 30°. is equal to the ratio of the opposite side to the hypotenuse. Therefore, by looking at the sketch sin 30° =1/2

Example 2. Evaluate cos 60° and cos 30°
Solution:
Step 1. Sketch a triangle and label with ratio numbers.



Step 2. We know that cos 60° and cos 30° are equal to the ratio of the adjacent side for each angle to the hypotenuse. Therefore, by looking at the sketch:

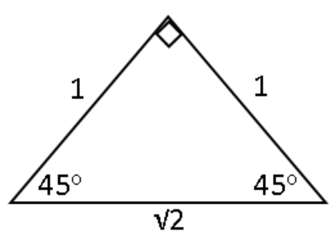
cos 60° =1/2

cos 30° =√3/2

Example 3. Evaluate sin 45°

Solution:

Step 1. Sketch a triangle and label ratio numbers.



Step 2. As before, we know sine of an angle is equal to the ratio of the opposite side to the hypotenuse.

Sin 45°= 1/√2, here we need to rationalize the denominator by multiplying both the numerator and the denominator by √2

Sin 45°= 1/√2=$\frac{\sqrt{2}}{\sqrt{2}\times\sqrt{2}}$ =$\frac{\sqrt{2}}{2}$

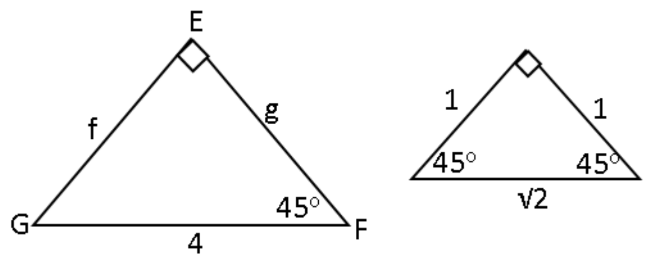
Practice problems:

Problem 1. Evaluate tan 30° and cot 60°

Problem 2. Evaluate cos 45^o and tan 45^o

Problem 3. Evaluate sec 30^o and csc 60^o

Problem 4. Solve the isosceles right triangle EFG given the hypotenuse GF=4 cm and the EFG=45^o .



Solution:

Step 1. To solve a triangle means to find the length of the sides and the measures of the angles. In the triangle EFG, angle EGF=180^o -90^o-45^o =45^o. Therefore, the EFG triangle is a 45^o -45^o -90^o isosceles triangle.

Step 2. The two triangles are similar because of equal angles. Therefore, the sides of two triangles are proportional. We know in 45^o -45^o -90^o triangle the sides are in ratio 1: √2: 1. The ratio of the hypotenuse of the triangle on the left to the hypotenuse of the triangle on the is 4√2 which is equivalent of 2√2 times (4/√2= (4 × √2)/(√2 × √2)=2√2). By proportionality rule, sides f=g=2√2×1=2√2

Geometry:

Geometry from Greek words "geo" means earth and "metria" means measure that has been used in different cultures for navigation and spatial relationships for centuries. Euclidean geometry is a field of mathematics that involves drawing and learning properties of two-dimensional or 2-D shapes (triangle, circle, rectangle…) and three-dimensional or 3-D shapes (cube, sphere, cylinder, …). These two parts of geometry are called **plane geometry** and **solid geometry**, respectively.

Plane geometry (2-D shapes):

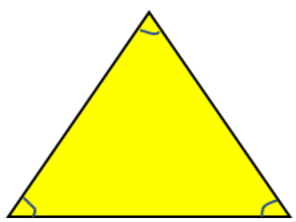
Triangles:

In geometry triangles are 2-D shapes with three sides and three angles that add to 180^o.

Types of triangles based on the number of equal sides:

Equilateral triangles:

Three equal sides and three equal angles (60^o)

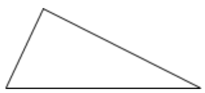


Isosceles triangles:

Two equal sides and two equal angles

Scalene triangles:

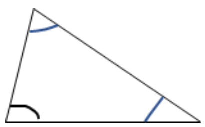
No equal sides and no equal angle



Types of triangles based on the type of angle:

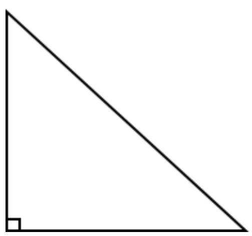
Acute triangle:

All angles are less than 90^o



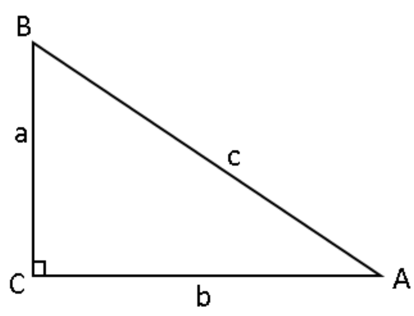
Right triangle:

Has a right angle (90^o)



The Pythagorean theorem:

This is a well-known concept the states that in right triangle, the square of the hypotenuse is equal to sum the squares of the other sides. For example, in the following right triangle ABC, $c^2=a^2+b^2$.



Example. Lis has recently purchased a 22 inch computer monitor with some specifications such as monitor's (width/height)=16/9. What is the measure of monitor's width?

Solution:

Step 1. Let's assume: x=height, then $x(\frac{16}{9})$=width

Step 2. Use the $c^2=a^2+b^2$ formula

$(22)^2=(\frac{16}{9}x)^2+(x)^2$,

$484=\frac{256}{81}x^2+x^2$.

Step 3. Add like variables, multiply both sides by 81, and take square root of both sides to find the value for x.

$484=\frac{337}{81}x^2$, $x^2=\frac{484\times81}{337}=116.33$, x=10.8 monitor's height

Step 4. Width$=\frac{16}{9}x=\frac{16}{9}\times10.8=19.2$

Mid-point and distance formula:

Distance formula $\mathbf{d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}$ is used to calculate the distance "d" between two points $A(x_1,y_1)$ and $B(x_2,y_2)$ in the coordinate plane. And mid-point formula, $\mathbf{m=((x_2+x_1)/2,\ (y_2+y_1)/2)}$ is the point in the middle of a line with endpoints $A(x_1,y_1)$ and $B(x_2,y_2)$.

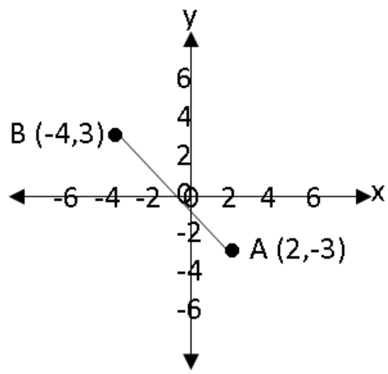
Example. In the graph below, the coordinate points A(2,-3) and B(-4,3) are given. Find the length of

Line AB and the mid-point of this line.

Solution: plug in the values into the formula

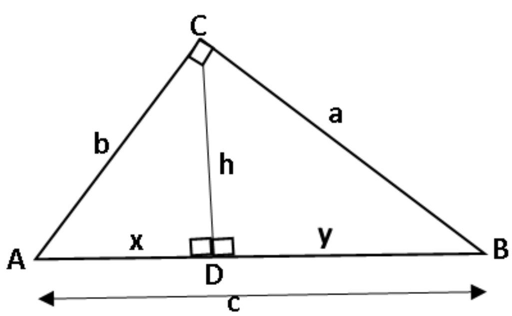
$d=\sqrt{(-4-2)^2+(-3-3_1)^2}=\sqrt{(36+36)}=\sqrt{72}=8.5$

$m=((-4+2)/2,(3-3)/2)=(-1,0)$



Altitude on hypotenuse theorem:

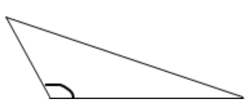
If an altitude is drawn to the hypotenuse of a right triangle as shown below, then



$h^2=xy$, $a^2=yc$, $b^2=xc$, and triangles ABC, ACD, BCD are similar

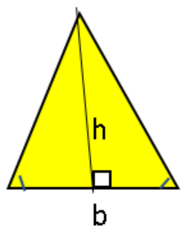
Obtuse triangle:

Has an angle more than 90^o



Area of a triangle:

A=1/2bh where h represents the height or altitude (perpendicular line from a vertex opposite it intersects) and b is the base (see below).



Quadrilaterals:

As the word quadrilateral implies (quad means four and lateral means side), they are four-sided shapes in geometry. Besides four sides, they have four vertices (corners) and four angles that add to 360^o.

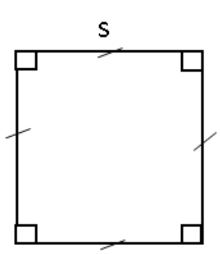
Types of quadrilaterals:

Square:

A. Four equal sides, four right angles, parallel opposite sides

B. Similar to rectangle for having four right angles and also similar to rhombus for having equal sides

C. The only regular quadrilateral (four equal sides and four equal angles)



Area of a square:

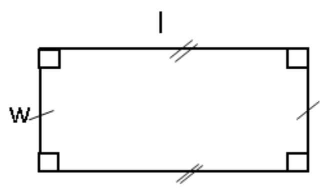
$\mathbf{A=s^2}$ where s is the side of the square

Rectangle (parallelogram):

A. Four right angles

B. Opposite sides are equal and parallel

C. Diagonals are equal in length and bisect each other



Area of a rectangle:

A=lw where is the length and is the width (see above)

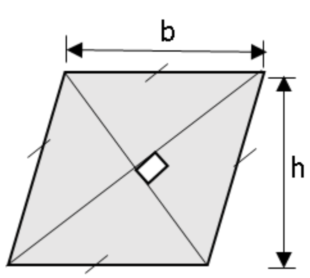
Rhombus:

A. All sides are equal

B. Opposite sides parallel

C. Opposite angles are equal

D. Diagonals are right angle bisectors of each other



Parallelogram:

A. Opposite sides are equal and parallel

B. Opposite angles are equal

C. Rectangle, square, and rhombus are parallelograms

D. Diagonals bisect each other

Area of a parallelogram:

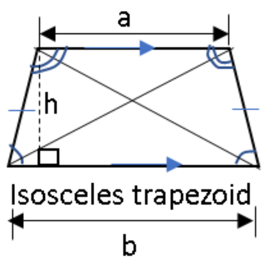
A=bh where b is the base and h is the altitude or height

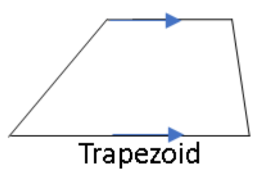
Trapezoid:

A. Has a pair of parallel sides

B. Isosceles trapezoid also has one pair of equal side that form two pairs of equal angles at the ends of one pair of equal sides. Diagonals are equal in isosceles trapezoid.

C. Trapezium has NO parallel sides





Area of trapezoid:

A=$\frac{a+b}{2}$ ×h where a and b are the lengths of the parallel sides and h is the height

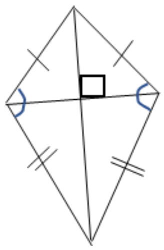
Kite:

A. Two pairs off equal sides

B. The two pairs of sides meet and form equal angles (angle between unequal sides)

C. One of the diagonals bisects the other at right angle

D. One diagonal bisects a pair opposite angles



Area of kite:

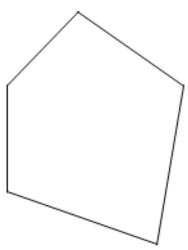
A=$\frac{pq}{2}$ where p and q are the lengths of the diagonals

Polygons:

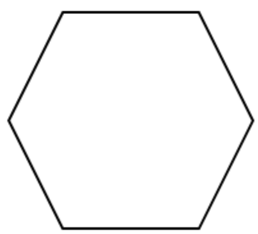
The word polygon is from “poly” means many and “gon” angle, is a 2-D geometric shape with many angles and many sides. In elementary geometry, polygon shapes are formed by a number of straight line segments coming together to form a closed shape. The points that two line-segments (sides or edges) meet to make a closed structure are called vertices (singular vertex) or corners. An n-gon is a polygon with n sides; for example, tetragon is a 4-sided polygon (tetra is a Greek word for four). Triangles are the examples for the simplest polygons. There are different types of Polygons.

1. Regular (equal sides and angles) or irregular
2. Convex (all interior angles are less than 180o) or concave
3. Simple or complex (where the polygon self-intersects).

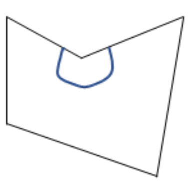
Remember that mathematicians are primarily concerned with closed chain of polygonal line segments and simple polygons. Here, our focus will be on simple and closed polygons and it shall be the focus student's learning as well. Each polygon has many vertices as it has sides. There are several angles around each vertex. These angles interior or exterior angles. The sum of the interior angles of a n-gon is **$(n-2)\times 180^{o}$ or $(n-2)\pi$ radians** . And the measure of any interior angle of convex regular n-gom is **$(1-2/n)\ \pi$ radians or $180-360/n$** degrees. This is because each simple n-gon can be divided into (n-2) triangles with angle sumof180 degrees for each triangle. Exterior angles are supplementary to the interior angles. A polygon diagonal is a line segment connecting two vertices. The number of the diagonals for a n-gon is **$n(n-3)/2$**. Two polygons are considered if all the angles are equal and also all the proportions of the sides are equal.



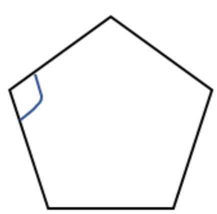
Irregular polygon



Regular polygon



Concave polygon containing an angle greater than 180°



convex polygon containing all angles less than 180°

Circles:

Definitions:

A set of points in plane equidistant from a central point called the center of the circle.

Radius:

a line segment from the center to a point on th e circle;

Circumference:

Distance around the edge of the circle.

Congruent circles:

Two circles with the same radius.

Diameter:

A line segment passing through the center and connecting two points on the edge of the circle.

Chord:

A line segment that connects two points on the circumference of the circle (does NOT pass through the center).

Tangent:

A line that intersects with a circle at only one point. The **radius** a t the point o tangency is **perpendicular** to the tangent line.

Secant:

A line that intersects with a circle at two points.

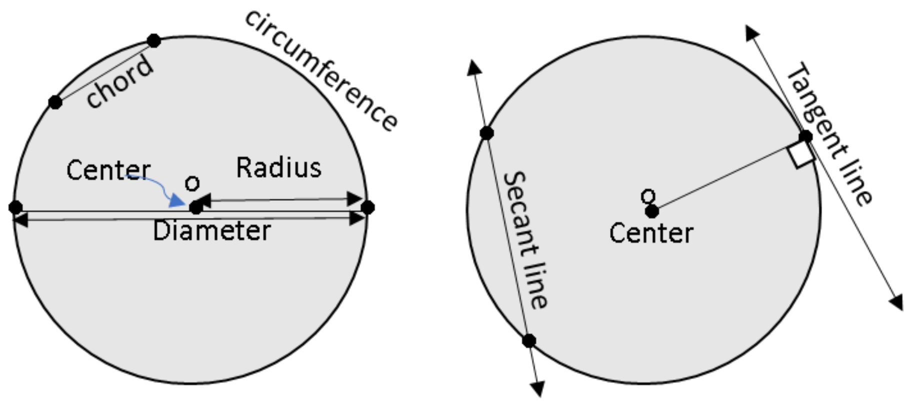
Inscribed angle:

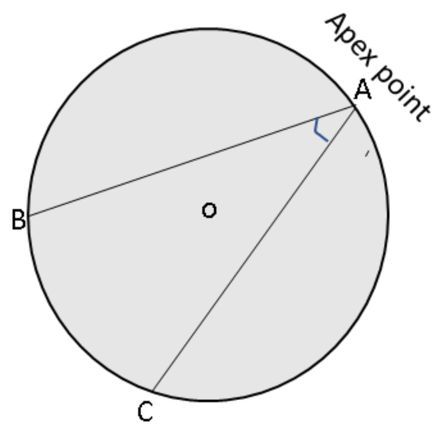
An angle made from points on the circumference of a circle.

Central angle:

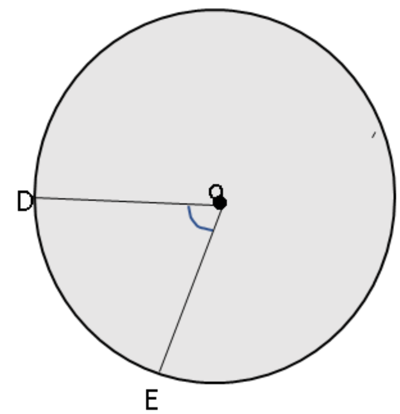
An angle with vertex at the center of the circle.

Arc. A part of the circle's circumference and has two measures; the length of the arc and the angle of the arc. **The measure of an arc** is **equal** to the central angle forming the arc and is named by its end-points such as arc DE or ED (see below). Using two letters to name an arc is confusing since it does not specify major or minor arc; for instance, we have major (going long way around the circle) or minor (shorter distance) arc named DE. To avoid confusion, I always use a point within the desired arc and name the arc with three points.





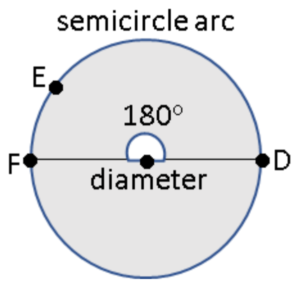
Inscribed angle with the end points B and C and the apex point A,



Central angle and the arc DE or ED

Semicircle arc.

This is when the measure of the arc is exactly half of the circumference. Notice in the figure below, the semicircle FED arc is specified by using three points F, E, and D instead of two points F and D.

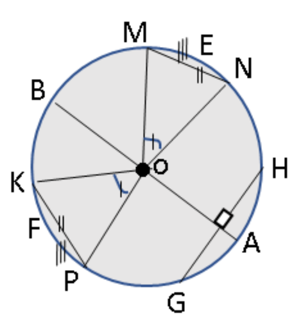


Chord central angle, chord arcs, and perpendicular line to a chord:

1. If two chords in a circle are congruent, the arcs formed by their intersections with the circle are congruent. The arcs KFP and MEN are congruent because the chords KP and MN are congruent (see below).

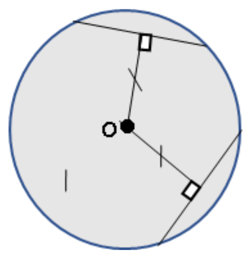
2. If two arcs are congruent, the central angles formed using the chords' intersection points on the circle as the end points of the angles are congruent. The central angles KOP and MON are congruent because the arcs KFP and MEN are congruent (see below).

3. The perpendicular line to a chord in a circle passes through the center of the circle and is the bisector of the chord. The perpendicular line BA is passing through the center (o) and bisects the chord GH (see below).



Chord distance :

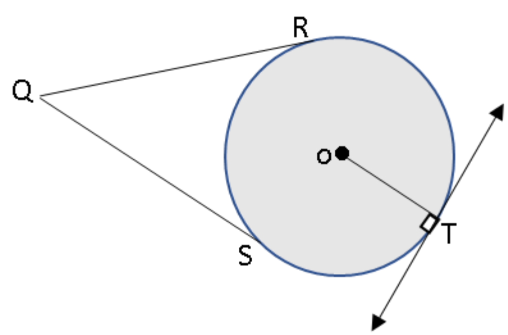
Two congruent circles chords in a circle are equal distant from the center of the center of the circle.



Tangent and tangent segment:

1. The tangent line to a circle is perpendicular to the radius at the point of tangency. The radius OT is perpendicular to the tangent line at the point of tangency T (see below)

2. Tangent segments from a point outside the circle are congruent. The tangents QR and QS are congruent.

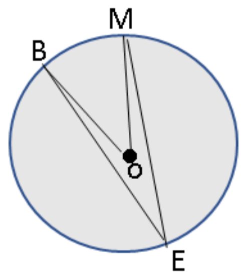


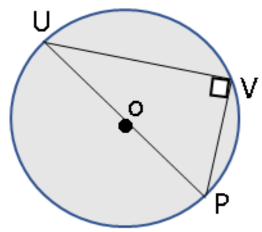
Inscribed angles, inscribed angles intercepting arcs, and angles inscribed in a semicircle:

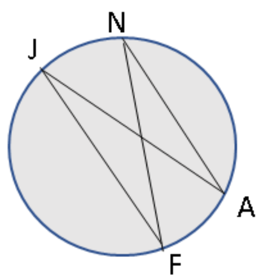
1. The measure of an inscribed angle is one-half the measure of the central angle in a circle. The measure of the BEM angle is one-half the measure of the BOM angle (see below).

2. Inscribed angles intercepting the same arc are congruent   The JFN angle and JAN angle are congruent (see below).

3. Inscribed angles in a semicircle are right angles. The UVP inscribed angle is a right angle with end points, U and P, on the diameter.

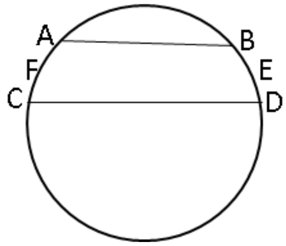






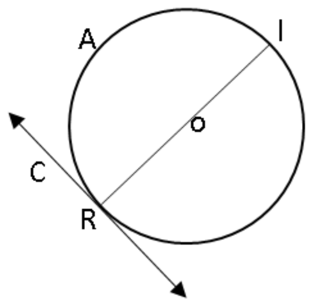
Parallel lines intercept congruent arcs:

The intercepted arcs AFC and BED by parallel lines AB and CD are congruent (see the diagram below).



Tangent and chord angle:

The measure of the angle formed by a tangent and a chord at the point of tangency is equal to one-half of the arc. Below the measure of the angle ORC is one-half the measure of the arc LAR.

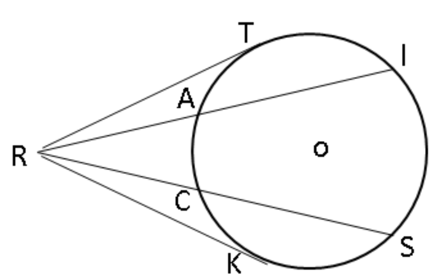


Two secants, two tangents, or tangent and secant angles:

1. The measure of the angle formed by two secant outside a circle is equal to one-half the measure of the difference of the intercepted arcs. The measure of the angle SRL is equal to one—half the measure of (arc LS - AC) (see figure below).

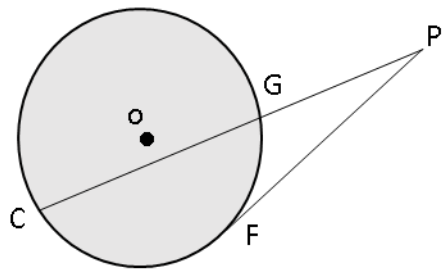
2. The measure of the angle formed by a tangent line and a secant outside a circle is equal to one-half of the measure of the difference of the intercepted arcs. Angle SRK=1/2 (arc SK – arc CK) (see below).

3. The measure of an angle formed between two tangents outside a circle is equal to the measure of the difference of the major and minor arcs. Angle KRT=1/2 (arc TLSK - KCAT)(see below).



Tangent and secant segment length:

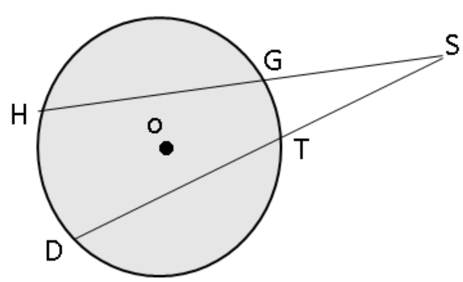
If a tangent and a secant are drawn from a point outside a circle, the square of the length of the tangent is equal to the product of the length of the secant and the external length of the secant. In the following diagram, $(PF)^2=PC \times PG$



Secant segments from point outside the circle:

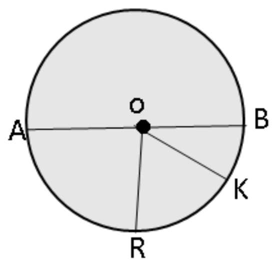
If two secants are drawn to a circle from a point outside the circle, the product of each secant to its external segment equals the product of the secant and its external segment. Below,

$SH \times SG = SD \times ST$.



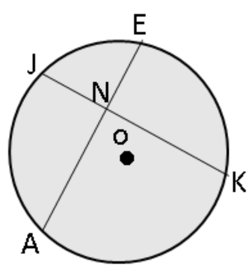
Correlation of the central angle and its arc

In a circle or two congruent circles containing two unequal central angles, the larger angle corresponds to larger arc and vice versa. Below, the measure of the central angle AOR is greater than the measure of the central angle BOK. Correspondingly, the measure of the related arc AB is larger the measure of the arc BK;



Intersecting chords within a circle:

When two chords intersect inside a circle, the product of the segments of one chord equals the product of the segments of the other chord. $AN \times NE = KN \times NJ$ (see below).



Area of a circle:

The area of a circle is equal to the product of the Greek letter π (represents a constant of 3.14159) and square of the radius (r^2).

$$A = \pi r^2$$

The circumference or the distance around circle is:

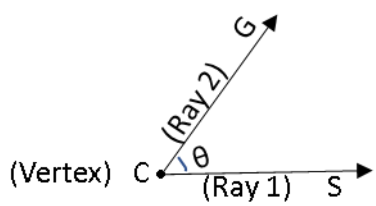
$$C = 2\pi r$$

Angle types and angle pairs:

An angle is formed by either the intersection of two rays in a plane (2D angles) or by the intersection of two planes in Euclidean and other spaces. Angles are commonly identified by either Greek letters α,β, γ, and θ. Angles are also identified by using the labels representing three points on defining the angles. For example, the angle θ shown below is also read as angle GCS.

2D angles:

The 2D angles are formed when two rays (will be the "sides" of the angle) intersect in a plane at a point called the vertex of the angle.



A polygon has one interior angle for each vertex and is formed by two adjacent sides. In a convex polygon, every internal angle is less than 180^o. A polygon containing any angle greater is called a non-convex or concave polygon. In contrast, exterior angle of a simple polygon is formed by extension of only one side at the vertex and is called vertical angle (two equal external angles for two extended sides for each vertex). The sum of the interior angles of a polygon with n sides is **180(n-2)^o** and the exterior angle sum of a simple polygon is 360^o .

Types of angles:

1. Acute angle. An angle less 90^o or smaller than a right angle;

2. Right angle. An angle equal to ¼ circle turn, 90^o, or π/2 radians. The two lines forming the right are called perpendicular, orthogonal or normal lunes.

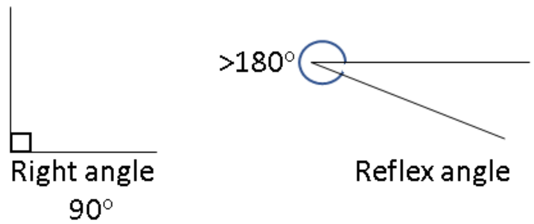
3. Obtuse angle. An angle larger than a right angle and smaller than straight angle (90^o-180^o).

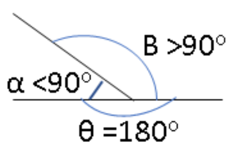
4. Straight angle. An angle equal to ½ circle turn,190^o, or 2π radians.

5. Reflex angle. An angle larger than 180° and smaller than 1 turn or 360°.

6. Perigon, full angle, or complete angle. An angle equal to 1turn, 360° or 2π.

7. Oblique angle. An angle that is NOT a right angle or a multiple of a right angle.





Straight (θ), acute (α), and obtuse (B) angles.

Angle pairs:

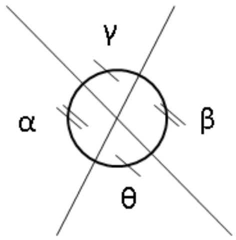
1. Congruent or equal angles. Angles with the same measure or magnitude.

2. Coterminal angles. Angles sharing terminal side but differing in size.

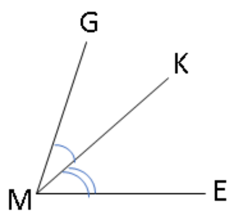
3. Reference angles. An acute angle determined by adding and subtracting straight angle or 180 degrees to the desired angle until the result is an acute angle. For instance, the reference angle for angles 60 and 165 degrees are 60 ° and (180-165=15°), respectively.

4. Opposite angles. When two straight lines intersect, the resulting two pairs of opposite angles are equal (vertically opposite angles θ and γ as well as opposite angles α and β). Let's look at the following diagram. When two adjacent angles such as α and γ are make a straight line on one side, they are supplementary (the angle sum for adjacent equals 180°). Therefore, $\alpha=180-\gamma$. Similarly, $\beta=180-\gamma$. consequently, $\alpha=\beta$. Also, angle θ is supplementary to angle α and angle β. Therefore, $\theta=180-(180-\gamma)$ and, in consequence, $\theta=\gamma$. Therefore, two pairs of opposite angles formed by two intersecting straight lines (see below) are equal

(**vertically opposite angles θ= γ and opposite angles α=β**).



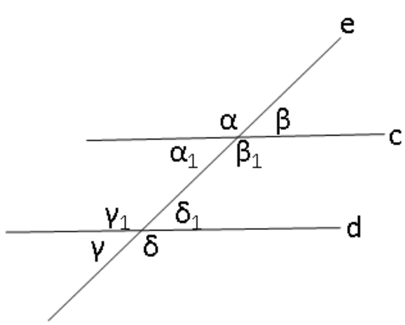
1. Adjacent angles. As the name implies, these angles are side by side or adjacent and share the vertex as well as a side (Angles GMK and KME below).



Adjacent angles whose sum is a straight angle, a right angle, or full angle are special angle pairs and called supplementary, complementary, or explementary angles, respectively.

2. Alternate interior, interior, exterior, and corresponding angles are angles formed by intersection of a transversal line and a pair of mostly parallel lines. Several congruent and supplementary angles are formed when a transversal (line "e" below) intersects two parallel lines. Four of these angles are exterior (α, β, γ, and δ) and the other four are interior (α_1, β_1, γ_1, and δ_1) angles. A pair of angles are said to be **corresponding angles if one angle** is interior and the other is exterior, have distinct vertex points, and both angles are on one side of the transversal. The angle pairs (α_1= γ), (γ_1= α), (δ=β_1), and (δ_1= β) are equal corresponding angles (see below). Using the Euclid's parallel postulate, we can state that if one pair of corresponding angles are equal, the other pairs of corresponding angles are equal. By a similar logic, if a pair of corresponding les formed by a transversal are equal then, the two lines transected by the transversal are parallel. **Alternate angles** are four pairs of angles that are on

opposite side of the transversal, both angles are interior or both angles are exterior, and have distinct vertex points. The exterior alternate angles pairs ($\alpha = \delta$) and ($\gamma = \beta$) resulting from intersection of the transversal "e" with two parallel lines are equal (see below). Also, interior alternate angles ($\alpha_1 = \delta_1$) and ($\gamma_1 = \beta_1$) Euclidean postulate states that if one pair of alternate angles are equal then, the two lines intersecting with the transversal are parallel and vice versa. And, if one pair of alternate angles are equal then, the other pairs of alternate angles are equal. **Consecutive interior angles** are two pairs of angles that are on one side of the transversal, have distinct vertex points, and both are interior. The two pairs of consecutive interior angles (α_1 and γ_1) and (β_1 and δ_1) are supplementary (sum to 180^o)(see below). alternatively, if any pair of consecutive interior angles are supplementary then, the two intersected lines by transversal are parallel. Also, if one pair of the pair is supplementary, the other one is also. A transversal that intersects two parallel lines in a right angle is called a perpendicular transversal and all eight angles formed are right angles.



Solid geometry (3D-shapes):

In geometry, solids are either polyhedrons (flat face) or non-polyhedrons (has non-flat face) 3 shapes (height, width, and depth). The word polyhedron (plural polyhedra or polyhedrons) comes from "poly" means "many" and "hedron" means "face". Examples of solid polyhedrons are cubes, pyramids, and cuboids. Solids such as cylinders, cones, and spheres are examples of non-polyhedrons. Polyhedra may me named by the number of the faces such as tetrahedron (4), pentahedron (5), hexahedron (6), heptahedron (7), octahedron (8), nonahedron (9), decahedron (10), and so on. Polyhedrons' volume, the number of faces, the number of edges and vertices, as well as the surface area are studied to determine their specific properties. In polyhedral the Euler formula features relationship of the number vertices (V),the number of faces, and the number of edges which is 2 in simple polyhedrons as **F+V-E=2** . Below you find the volume for some polyhedrons.

Shape	Volume	Variable
prism	Ah	A=bae area, h=height
sphere	$4/3\pi r^3$	r=radius
Cylinder	$\pi r^2 h$	r=base radius h=height
cube	s^3	S=side
cone	$1/3\pi r^2 h$	R=base radius, h=height
Rectangular pyramid	1/3lwh	L=length, w=width, h=height
pyramid	1/3ah	A=base area, h=height