Grade 12 Mathematics

Chapter 1) Matrices

We need to review the matrices before discussing the vectors. A **n×m** matrix is a matrix with n rows and m columns where **n × m** is the size or **dimension** of the matrix and $\boldsymbol{k_{ij}}$ denotes. the entry in the i^{th} row and j^{th} column generally written as follows:

$$K=\begin{bmatrix} k_{11} & k_{12} & \dots & k_{1m} \\ k_{21} & k_{22} & \dots & k_{2m} \\ . & . & & . \\ . & . & & . \\ k_{n1} & k_{n2} & \dots & k_{nm} \end{bmatrix}_{n\times m} = (k_{ij})_{n\times m}$$ where the subscripted n×m matrix is often dropped

A matrix is basically an arranged set of numbers in rows and columns represented usually within a square bracket as $\begin{bmatrix} 2 & 1 & -3 \\ -4 & 0 & 5 \end{bmatrix}$ with two rows and three columns. The five major special matrices we review here are zero, identity, and square, column, and row matrices. The square matrices have the same number of rows and columns. The identity matrix (represented as I_n) is a square matrix whose main diagonal (a diagonal in the square matrix that starts in the upper left and ends in the lower tight) is all 1's and whose all of the other elements are zeroes. The zero matrix (denoted as $0_{n\times m}$) is a matrix whose all of its entries are zeroes.

$$I_n=\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ . & . & & . \\ . & . & & . \\ 0 & 0 & \dots & 1 \end{bmatrix}_{n\times m} \qquad 0_{n\times m}=\begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ . & . & & . \\ . & . & & . \\ 0 & 0 & \dots & 0 \end{bmatrix}_{n\times m}$$

A column or row matrix is a matrix with a single column or a single row, respectively, as shown below.

K= $\begin{bmatrix} k_{11} \\ k_{21} \\ . \\ . \\ k_{n1} \end{bmatrix}_{n\times 1}$ K=$[k_1 \quad k_1 \ldots \; k_m]_{1\times m}$

The **addition and subtraction** of two matrices with equal size, $X_{n\times m}$ and $Y_{n\times m}$ is as follows:

$X_{n\times m} \pm Y_{n\times m}$ = $(x_{ij} \;\pm y_{ij})_{n\times m}$ = $(x_{ij})_{n\times m} \pm (y_{ij})_{n\times m}$

The new matrix after addition or subtraction is a matrix with the same size and the entries is the sum or difference of the corresponding entries from the original two matrices., respectively. Remember that we can't add or subtract entries from matrices with sizes. The product of constant c and matrix X, **scalar multiplication**, is as follows:

c $X_{n\times m}$ = c$(x_{ij})_{n\times m}$ = $(cx_{ij})_{n\times m}$

Example. What is the result of 2X-4Y for the following matrices:

X= $\begin{bmatrix} -2 & 5 \\ -1 & 6 \end{bmatrix}$ and **Y=** $\begin{bmatrix} -6 & -4 \\ -2 & 6 \end{bmatrix}$

Solution:

2X=2 $\begin{bmatrix} -2 & 5 \\ -1 & 6 \end{bmatrix}$**=**$\begin{bmatrix} -4 & 10 \\ -2 & 12 \end{bmatrix}$

4Y= 4 $\begin{bmatrix} -6 & -4 \\ -2 & 6 \end{bmatrix}$ **=** $\begin{bmatrix} -24 & -16 \\ -8 & 24 \end{bmatrix}$

2X-4Y=$\begin{bmatrix} -4 & 10 \\ -2 & 12 \end{bmatrix}$**-**$\begin{bmatrix} -24 & -16 \\ -8 & 24 \end{bmatrix}$ **=** $\begin{bmatrix} 20 & 26 \\ 6 & -12 \end{bmatrix}$**,** so, each entry of the matrix X is multiplied by 2 and form a new matric 2X. likewise, each entry of the matrix Y is multiplied by 4 and form a new matrix 4Y. Then, the corresponding entries from the 4Y matrix is subtracted from the matrix 2X and form a new matrix 2X-4Y.

Now, we need to take a look at is the **matrix multiplication**. The product of matrix $X_{l\times m}$ and $Y_{m\times o}$ is as follows, notice that based on the current convention of matrix operation, the multiplication (X×Y) is **true or defined only when the number columns of the X matrix is equal to the number of rows of the Y matrix, otherwise it is undefined**. Also, order matters when multiplying the matrices:

$X_{l\times m}\, Y_{m\times o}=\left[a_{ij}\right]_{lo}$ where the number of columns of the matrix X must be the same as the number of the rows of the matrix Y.

Example. Given the following matrices, what is AB?

A= $\begin{bmatrix} -1 & 2 & 3 \\ -2 & 4 & 0 \end{bmatrix}_{2\times 3}$ B= $\begin{bmatrix} 0 & 2 & 4 \\ 1 & -1 & 3 \\ 0 & 5 & -2 \end{bmatrix}_{3\times 3}$

Solution:

AB= $\begin{bmatrix} -1 & 2 & 3 \\ -2 & 4 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 2 & 4 \\ 1 & -1 & 3 \\ 0 & 5 & -2 \end{bmatrix}$

The matrix product is computed as follows. Let $\boldsymbol{c_{ij}}$ denotes the entries for the new matrix.

c_{11}=(-1)(0)+(2)(1)+(3)(0)=2

c_{12}=(-1)(2)+(2)(-1)+(3)(5)=11

c_{13}=(-1)(4)+(2)(3)+(3)(-2)=-4

c_{21}=(-2)(0)+(4)(1)+(0)(0)=4

c_{22}=(-2)(2)+(4)(-1)+(0)(5)=-8

c_{23}=(-2)(4)+(4)(3)+(0)(-2)=4

The resultant 2×3 AB matrix is $\begin{bmatrix} 2 & 11 & -4 \\ 4 & -8 & 4 \end{bmatrix}_{2\times 3}$. Remember that matrix multiplication operation, **rows hit columns and fill up rows**.

Let's take a look at the entry c_{11}, for example. Simply, the first, second, and third entries of the first row from A are multiplied by the corresponding first, second, and third entries of B, respectively. Then, add them all up.

The division of two matrices X and Y is as follows:

X/Y=X$\frac{1}{Y}$ =XY^{-1} , so, as you note the division of X by Y is rewritten as X times the negative exponent of Y matrix. This means that we don't divide but multiply by an **inverse matrix** (Y^{-1}). We'll learn how find the inverse matrix later on.

The next topic we need to take a look at is the **determinant** of a matrix. The determinant of a matrix is special number computed from a square matrix. If the determinant of a matrix is zero, then the matrix is called **singular** matrix. For instance, the matrix A=$\begin{bmatrix} 3 & 6 \\ 4 & 8 \end{bmatrix}$ is a singular matrix for 3×8-6×4=0. A matrix with non-zero determinant is **nonsingular** matrix. The standard notation for the determinant of the matrix X is **det(X)=|X|**. The following formulas are used to calculate the determinant of 2×2 and 3×3 matrices.

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$. Determinant can be used to solve systems with the same number of equations as variables, in which the value of the variable of interest is determined by a quotient of two matrices. The numerator of this quotient is the matrix of coefficients after replacing the column of the variable of interest with the constant column. And the denominator is a matrix of coefficients. The determinant of these matrices will be used in the solution of the variable of interest. For example, consider the following system of general system of equations.

$$\begin{cases} a_1 x + b_1 y = c_1 \\ a_2 x + b_2 y = c_2 \end{cases}$$

$$x = \frac{\begin{matrix} c_1 & b_1 \\ c_2 & b_2 \end{matrix}}{\begin{matrix} a_1 & b_1 \\ a_2 & b_2 \end{matrix}} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$y = \frac{\begin{matrix} a_1 & c_1 \\ a_2 & c_2 \end{matrix}}{\begin{matrix} a_1 & b_1 \\ a_2 & b_2 \end{matrix}} = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Example: Find th e determinant for the following matrices.

C=$\begin{bmatrix} -1 & 6 \\ 4 & 5 \end{bmatrix}$ D=$\begin{bmatrix} 3 & -2 & 7 \\ 1 & 4 & 8 \\ 2 & 3 & -3 \end{bmatrix}$

Solution:

C=$\begin{bmatrix} -1 & 6 \\ 4 & 5 \end{bmatrix}$ is a 2×2 matrix and there isn't much to do other than using the given simple formula above.

det(C) =$\begin{vmatrix} -1 & 6 \\ 4 & 5 \end{vmatrix}$=$a_{11}a_{22}$-$a_{12}a_{21}$ =(-1)(5)-(6)(4)=-29

D=$\begin{bmatrix} 3 & -2 & 7 \\ 1 & 4 & 8 \\ 2 & 3 & -3 \end{bmatrix}$ is a 3×3 matrix. we'll use two methods to find the determinant for this matrix.

Method 1. Use the given formula above.

det(D)= $\begin{vmatrix} 3 & -2 & 7 \\ 1 & 4 & 8 \\ 2 & 3 & -3 \end{vmatrix}$=$a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$-$a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$+$a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$=

3$\begin{vmatrix} 4 & 8 \\ 3 & -3 \end{vmatrix}$-(-2)$\begin{vmatrix} 1 & 8 \\ 2 & -3 \end{vmatrix}$+7$\begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix}$=

3$[(4)(-3)-(8)(3)]$-(-2)$[(1)(-3)-(8)(2)]$+7$[(1)(3)-(4)(2)]$=

3(-36)-(-2)(-19)+7(-5)=-181

Method 2. Tack a copy of the first two columns onto the end of the matrix as follows.

det(D)= $\left|\begin{matrix} 3 & -2 & 7 \\ 1 & 4 & 8 \\ 2 & 3 & -3 \end{matrix}\right|\begin{matrix} 3 & -2 \\ 1 & 4 \\ 2 & 3 \end{matrix}$

multiply the entries of the diagonals from top right to bottom left, add them up. Then, subtract the result from the sum of the products of the entries of the diagonals from top left to bottom right

(3)(4)(-3)+(-2)(8)(2)+(7)(1)(3)-(-2)(1)(-3)-(3)(8)(3)-(7)(4)(2)=-181

Next, we need to take a look at what is called **inverse matrix**. B^{-1} is the denoted inverse matrix of B exists only when $\boldsymbol{B^{-1}}$**B=I(identity matrix)**. Recall that the identity matrix is a matrix in which all the entries of the main or principal diagonal (the diagonal containing $a_{11}, a_{22}, a_{33}, \ldots a_{nn}$) are ones and all other entries are zeroes, for example, I=$\begin{Bmatrix} 1 & 0 \\ 0 & 1 \end{Bmatrix}$ is a 2×2 identity matrix. For 2×2 square matrix A=$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the inverse matrix is computed as follows:

A^{-1}=$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$ =$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. it means swap the positions of a and d, put negative signs in front of b and c, divide the whole thing by the determinant.

Example 1. Find the inverse of the P=$\begin{bmatrix} 4 & 0 \\ 2 & 1 \end{bmatrix}$.

Solution:

Method 1. P^{-1}=$\begin{bmatrix} 4 & 0 \\ 2 & 1 \end{bmatrix}^{-1}$ = $\frac{1}{(4\times1)-(0\times2)}\begin{bmatrix} 1 & -0 \\ -2 & 4 \end{bmatrix}$= $\frac{1}{4}\begin{bmatrix} 1 & -0 \\ -2 & 4 \end{bmatrix}$= $\begin{bmatrix} 0.25 & 0 \\ -0.5 & 1 \end{bmatrix}$

If $\begin{bmatrix} 0.25 & 0 \\ -0.5 & 1 \end{bmatrix}$ is the inverse of the given function, then the following must be true.

$\begin{bmatrix} 4 & 0 \\ 2 & 1 \end{bmatrix}$. $\begin{bmatrix} 0.25 & 0 \\ -0.5 & 1 \end{bmatrix}$=$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ =I, so let's find out.

$\begin{bmatrix} 4\times0.25+0(-0.5) & 4\times0+0\times1 \\ 2\times0.25+1(-0.5) & 2\times0+1\times1 \end{bmatrix}$=$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, this is identity matrix and the inverse is true.

Method 2. We use the **elementary row operations** (used to solve systems) which is summarized as the followings:

1. multiply a row by a constant denoted cR_i

2. interchange rows denoted $R_i \leftrightarrow R_j$

3. add multiple of one row to another denoted $cR_i + R_j$

Add 2×2 identity on the given 2×2 matrix. Recall that here the goal os to convert the first two columns to 2×2 identity through applying the rules for row operations.

P=$\begin{bmatrix} 4 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \overset{-0.5R_1+R_2}{\rightarrow} \begin{bmatrix} 4 & 0 & 1 & 0 \\ 0 & 1 & -0.5 & 1 \end{bmatrix}$

$\overset{0.25R_1}{\rightarrow} \begin{bmatrix} 1 & 0 & 0.25 & \mathbf{0} \\ 0 & 1 & -0.5 & 1 \end{bmatrix}$

So, the first two columns is converted to identity, therefore, the inverse is the second two columns of the matrix; the 2×2 square matrix $\begin{bmatrix} 0.25 & 0 \\ -0.5 & 1 \end{bmatrix}$ is the inverse of the given matrix in question.

Example 2. Determine whether the inverse of the 3×3 matrix

Q=$\begin{bmatrix} 2 & -1 & 2 \\ 4 & 8 & -2 \\ -2 & 3 & 1 \end{bmatrix}$ exists?

Solution:

Add a 3×3 identity onto the given matrix

$\begin{bmatrix} 2 & -1 & 2 & 1 & 0 & 0 \\ 4 & 8 & -2 & 0 & 1 & 0 \\ -2 & 3 & 1 & 0 & 0 & 1 \end{bmatrix}$=

As mentioned above, we apply row operations until we get the first three columns becomes 3×3 identity

$\begin{matrix} 0.5R_1 \rightarrow \\ 2R_3 + R_2 \\ 2R_1 + R_3 \end{matrix} \begin{bmatrix} 1 & -0.5 & 1 & 0.5 & 0 & 0 \\ 0 & 14 & 0 & 0 & 1 & 2 \\ 0 & 2 & 3 & 1 & 0 & 1 \end{bmatrix}$=

$$\begin{matrix} 7R_3 \to \\ -R_2+R_3 \\ \frac{1}{14}R_2 \\ \frac{1}{21}R_3 \end{matrix} \begin{bmatrix} 1 & -0.5 & 1 & 0.5 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{14} & \frac{1}{14} \\ 0 & 0 & 1 & \frac{7}{21} & -\frac{1}{21} & \frac{5}{21} \end{bmatrix} =$$

$$\begin{matrix} \to \\ 0.5R_2+R_1 \\ -R_{3+}R_1 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & \frac{1}{6} & \frac{1}{12} & -\frac{17}{84} \\ 0 & 1 & 0 & 0 & \frac{1}{14} & \frac{1}{14} \\ 0 & 0 & 1 & \frac{7}{21} & -\frac{1}{21} & \frac{5}{21} \end{bmatrix}$$

So, the first three columns are changed to identity, therefore, the second three columns

$\begin{bmatrix} \frac{1}{6} & \frac{1}{12} & -\frac{17}{84} \\ 0 & \frac{1}{14} & \frac{1}{14} \\ \frac{7}{21} & -\frac{1}{21} & \frac{5}{21} \end{bmatrix}$ is the 3×3 inverse of the given 3×3 matrix.

Not all matrices have inverse. Consider the square matrix A=$\begin{bmatrix} 1 & -4 \\ -2 & 8 \end{bmatrix}$. We learned how to find the inverse of a 2×2 matrix; tentatively, it should A^{-1}=$\begin{bmatrix} 8 & 4 \\ 2 & 1 \end{bmatrix}$. However, if that is actually the inverse then AA^{-1}= $\begin{bmatrix} 1 & -4 \\ -2 & 8 \end{bmatrix}\begin{bmatrix} 8 & 4 \\ 2 & 1 \end{bmatrix}$=I. let's multiply the two matrices to determine whether it equals the identity.

$\begin{bmatrix} 1\times 8+(-4)2 & 1\times 4+(-4)1 \\ (-2)8+2\times 8 & (-2)4+8\times 1 \end{bmatrix}$=$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so the inverse for the given matrix doesn't exist as AA^{-1}= $\begin{bmatrix} 1 & -4 \\ -2 & 8 \end{bmatrix}\begin{bmatrix} 8 & 4 \\ 2 & 1 \end{bmatrix}$≠I.